Analytical Assessment of Multi-user vs. Spatial Diversity Trade-offs with Delayed Channel State Information

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***Abstract*— In this letter, we explore the combined use of spatial and multi-user diversity in a cellular system where channel state information is subject to delays in the feedback channel. First, we *analytically* derive the probability and cumulative density functions of the post-scheduling signal-to-noise ratio (SNR) for both a Single-Input Single-Output (SISO) and an Orthogonal Space-Time Block Coding (OSTBC) transmission schemes. Then, we obtain the *closed-form* expressions of the corresponding average system capacities. By evaluating those expressions, we *analytically* show that the OSTBC scheme is far less sensitive to delays in the feedback channel.**

***Index Terms*— Spatial diversity, multi-user diversity, delayed channel state information, space-time coding.**

I. INTRODUCTION

IN a wireless multi-user system, the average cell throughput can be substantially increased when in each time-slot the user with the most favorable channel conditions is scheduled [1]. This effect is referred to as multi-user diversity (MUD) and relies on the assumption that different users in the system experience independent fading processes. In such fading environments, the exploitation of transmit spatial diversity (e.g., by means of Orthogonal Space-Time Block Coding, OSTBC) makes transmission links more robust by using low-complexity receivers [2] and, for that reason, much attention has been recently paid to the combined use of multi-user and transmit spatial diversity. In [3], for instance, the authors show that in a multi-user context Single-Input Single-Output (SISO) schemes outperform OSTBC-based ones in terms of aggregated cell capacity. Certainly, spatial diversity helps reduce the probability of deep fades but, by averaging over different transmit diversity branches, signal-to-noise ratio (SNR) peaks (those that multi-user diversity can exploit) are suppressed as well; hence, the overall system capacity decreases. In [4], it was proven that with perfect Channel State Information (CSI) at the transmitter, spatial diversity can be efficiently exploited in a multi-user context by using optimal beamforming. Unfortunately, *perfect* CSI is seldom available at the base station. Alternatively, a scheme that concentrates all the power in the transmit antenna with largest gain is also considered. For this second approach only low-rate partial CSI is needed but its performance is considerably sensitive to imperfections in the feedback channel. Recently, several studies show that the increased robustness of OSTBC schemes against imperfect CSI provides significant capacity gains with respect to those of SISO approaches. For instance, the impact of delay was addressed in [5] but, in that occasion, this issue was exclusively assessed by means of system-level computer simulations. In this paper, we conduct an analytical study of the impact of delayed CSI on SISO and OSTBC-based schemes. To do that, we derive closed-form expressions of the post-scheduling SNR statistics and the average system capacity as a function of the feedback delay. We *analytically* show that using OSTBC pays off in some situations.

II. SIGNAL MODEL AND SCHEDULER

Consider the downlink of a cellular system with one Base Station (BS) equipped with M antennas, and K single-antenna Mobile Stations (MS). The received signal at the k-th mobile station is given by:

$$r_k = \mathbf{h}_k^T \mathbf{s} + n_k \tag{1}$$

where $\mathbf{h}_k \in \mathbb{C}^M$ is the channel vector gain between the BS and the k-th terminal, for which each component is assumed to be independent and identically distributed, circularly symmetric Gaussian random variable with zero mean and user-dependent variance $\sigma^2_{h_k}(\mathbf{h}_k \sim \mathcal{CN}(\mathbf{0}, \sigma^2_{h_k}\mathbf{I}_M))$, $\mathbf{s} \in \mathbb{C}^M$ denotes the symbol vector, and $n_k \in \mathbb{C}$ stands for sampled additive white Gaussian noise with zero mean and variance σ^2 (time index is dropped for brevity). Statistical independence is assumed for the channel gains corresponding to different transmit antennas and different users in the system. We denote by $\gamma_k = \frac{P_t\|\mathbf{h}_k\|^2}{M\sigma^2}$ the *instantaneous* SNR experienced by user k in a given time-slot and by $\bar{\gamma}_k = \frac{P_t \mathrm{E}[\|\mathbf{h}_k\|^2]}{M\sigma^2}$ its long-term average SNR, with P_t standing for the total transmit power. Notice that the total transmitted power is constant and evenly distributed among transmit antennas. At the BS, we will consider two transmission schemes: a SISO configuration ($M = 1$) and an OSTBC scheme with $M = 2$ transmit antennas[1]. As for the scheduling process, it is organized on a slot-by-slot basis following a modified version of the Proportional Fair Scheduling [6] rule. In particular, in each time slot the user with the largest normalized SNR, $\frac{\gamma_k}{\bar{\gamma}_k}$, is selected for transmission. By doing so, users are only allowed to transmit

[1]For simplicity, we have assumed only two transmit antennas but the analysis can be easily extended to the general case.

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when their instantaneous SNR is near *its own peak* and, as a result, multi-user diversity is still exploited by granting access probability of $1/K$ to each user. In the presence of delays in the return channel, though, one should notice that the partial CSI information available at the scheduler, $\hat{\gamma}_k$, differs from the actual SNR γ_k.

III. Derivation of the Post-Scheduling SNR Statistics

Since system capacity is closely related with the post-scheduling SNR, i.e., the SNR experienced by the scheduled user, we will start by deriving an expression of its probability density function (pdf). As a previous step, one should identify the pdf of the actual SNR, $\gamma_k = \frac{P_t\|\mathbf{h}_k\|^2}{M\sigma^2}$, conditioned on the (delayed) estimate available at the centralized scheduler, $\hat{\gamma}_k = \frac{P_t\|\hat{\mathbf{h}}_k\|^2}{M\sigma^2}$. Throughout this work, we will adopt a Jakes' scattering model and, hence, $\mathbf{h}_k$ and $\hat{\mathbf{h}}_k$ turn out to be samples of the same Gaussian process. In other words, $\mathbf{h}_k$ and $\hat{\mathbf{h}}_k$ follow a joint complex Gaussian distribution with correlation coefficient $\rho_k = J_o(2\pi f_{d_k} T_k)$, where f_{d_k} stands for the Doppler frequency, T_k is the delay in time units, and $J_o(\cdot)$ denotes the zero-order Bessel function of the first kind. As a consequence, the pdf of $\mathbf{h}_k$ conditioned on $\hat{\mathbf{h}}_k$ can be easily obtained by applying Bayes' Theorem:

$$\begin{aligned} f_{\mathbf{h}_k|\hat{\mathbf{h}}_k}(\mathbf{h}_k|\hat{\mathbf{h}}_k) &= \frac{f_{\mathbf{h}_k,\hat{\mathbf{h}}_k}(\mathbf{h}_k,\hat{\mathbf{h}}_k)}{f_{\hat{\mathbf{h}}_k}(\hat{\mathbf{h}}_k)} \\ &= \frac{1}{\pi^M \det(\mathbf{R}_k)} e^{-(\mathbf{h}_k-\rho_k\hat{\mathbf{h}}_k)^H \mathbf{R}_k^{-1}(\mathbf{h}_k-\rho_k\hat{\mathbf{h}}_k)} \end{aligned} \quad (2)$$

where $\mathbf{R}_k = (1-\rho_k^2)\mathbf{I}_M$ is the covariance matrix. Notice that $\mathbf{R}_k$ models the degree of CSI uncertainty and, hence, it is equal to $\mathbf{R}_k = \mathbf{0}$ when $\mathbf{h}_k = \hat{\mathbf{h}}_k$. From this expression it is straightforward to show that γ_k conditioned on its estimate, $\hat{\gamma}_k$, follows a non-central chi-square distribution with $2M$ degrees of freedom:

$$\begin{aligned} f_{\gamma_k|\hat{\gamma}_k}(\gamma_k|\hat{\gamma}_k) &= \frac{M}{\bar{\gamma}_k(1-\rho_k^2)} \left(\frac{\gamma_k}{\rho_k^2\hat{\gamma}_k}\right)^{\frac{2M-2}{4}} e^{\frac{-M(\gamma_k+\rho_k^2\hat{\gamma}_k)}{\bar{\gamma}_k(1-\rho_k^2)}} \\ &\quad \times I_{M-1}\left(\frac{2M\sqrt{\rho_k^2\gamma_k\hat{\gamma}_k}}{\bar{\gamma}_k(1-\rho_k^2)}\right) \end{aligned} \quad (3)$$

with $I_n(\cdot)$ standing for the nth-order modified Bessel function of the first kind. Now, by defining A_i as the event that user i is the selected user, $A_i = \{\frac{\hat{\gamma}_i}{\bar{\gamma}_i} = \max_{k=1..K}\frac{\hat{\gamma}_k}{\bar{\gamma}_k}\}$, one can readily obtain the CDF of the post-scheduling SNR, γ^*, as:

$$\begin{aligned} F_{\gamma^*}(y) &= \sum_{i=1}^{K} \text{Prob}(\gamma_i \le y|A_i)\,\text{Prob}(A_i) \\ &= \sum_{i=1}^{K}\int_0^\infty F_{\gamma_i|\hat{\gamma}_i}(y|\hat{\gamma}_i) f_{\hat{\gamma}_i|A_i}(\hat{\gamma}_i|A_i)d\hat{\gamma}_i \text{Prob}(A_i) \\ &= \frac{1}{K}\sum_{i=1}^{K}\int_{\gamma_i=0}^{y}\int_{\hat{\gamma}_i=0}^{\infty} f_{\gamma_i|\hat{\gamma}_i}(\gamma_i|\hat{\gamma}_i) f_{\hat{\gamma}_i|A_i}(\hat{\gamma}_i|A_i)d\gamma_i d\hat{\gamma}_i \end{aligned} \quad (4)$$

where $\text{Prob}(A_i) = 1/K$, $i = 1..K$, due to the scheduler properties and $f_{\hat{\gamma}_i|A_i}(\hat{\gamma}_i|A_i)$ denotes the pdf of the *estimated* post-scheduling SNR associated to user i conditioned on event A_i. In such a multi-user context with independently distributed Rayleigh fading channels and a max-normalized SNR scheduler, we can write $f_{\hat{\gamma}_i|A_i}(\hat{\gamma}_i|A_i)$ as [7]:

$$f_{\hat{\gamma}_i|A_i}(\hat{\gamma}_i|A_i) = \frac{K}{\bar{\gamma}_i} f_i'\left(\frac{\hat{\gamma}_i}{\bar{\gamma}_i}\right)\prod_{\substack{k=1\\k\neq i}}^{K} F_k'\left(\frac{\hat{\gamma}_i}{\bar{\gamma}_i}\right) \quad (5)$$

where $f_k'(\cdot)$ and $F_k'(\cdot)$ are the pdf and CDF of the normalized SNR associated to user k, $\frac{\hat{\gamma}_k}{\bar{\gamma}_k}$. Then, by bearing in mind that $F_k'(x)$ is equal to $1-e^{-x}$ and $1-e^{-2x}(2x+1)$ for the SISO and OSTBC configurations respectively, we can particularize the above result to the different transmission schemes:

$$f_{\hat{\gamma}_{SISO_i}|A_i}(\hat{\gamma}_i|A_i) = K\frac{e^{-\frac{\hat{\gamma}_i}{\bar{\gamma}_i}}}{\bar{\gamma}_i}\left(1-e^{-\frac{\hat{\gamma}_i}{\bar{\gamma}_i}}\right)^{K-1} \quad (6)$$

$$f_{\hat{\gamma}_{OSTBC_i}|A_i}(\hat{\gamma}_i|A_i) = \frac{4K\hat{\gamma}_i e^{-\frac{2\hat{\gamma}_i}{\bar{\gamma}_i}}}{\bar{\gamma}_i^2}\left(1-e^{-\frac{2\hat{\gamma}_i}{\bar{\gamma}_i}}\left(\frac{2\hat{\gamma}_i}{\bar{\gamma}_i}+1\right)\right)^{K-1} \quad (7)$$

Last, by plugging (6) (or (7)) along with (3) into (4), the corresponding post-scheduling CDF expressions can be obtained. For the SISO approach, by using the binomial expansion and identities Eq. 6.614.3, Eq. 9.220.2 and Eq. 9.215.1 in [8] one can show that:

$$F_{\gamma^*_{SISO}}(y) = \sum_{i=1}^{K}\sum_{k=0}^{K-1}\binom{K-1}{k}\frac{(-1)^k}{k+1}\left(1-e^{-\frac{y(k+1)}{\bar{\gamma}_i(1+(1-\rho_i^2)k)}}\right) \quad (8)$$

On the other hand, for the OSTBC case one should resort to identities Eq. 8.406.3, Eq. 6.643.4 and Eq. 8.970.1 in [8] instead and, then, the CDF can be expressed in terms of the incomplete gamma function ($\Gamma(n,x) = \int_0^x e^{-t}t^{n-1}dt$ [8]):

$$\begin{aligned} F_{\gamma^*_{OSTBC}}(y) &= \sum_{i=1}^{K}\sum_{k=0}^{K-1}\binom{K-1}{k}(-1)^k\sum_{n=0}^{k}\binom{k}{n}\sum_{m=0}^{n}\binom{n+1}{n-m} \\ &\quad \times \frac{n!(1-\rho_i^2)^{n-m}\rho_i^{2m}\Gamma\left(m+2, \frac{2y(k+1)}{\bar{\gamma}_i(1+(1-\rho_i^2)k)}\right)}{m!(k+1)^{m+2}(1+(1-\rho_i^2)k)^n} \end{aligned} \quad (9)$$

Notice that, by solving only the inner integrals in equation (4), the corresponding post-scheduling pdfs can be obtained:

$$f_{\gamma^*_{SISO}}(\gamma) = \sum_{i=1}^{K}\sum_{k=0}^{K-1}\binom{K-1}{k}\frac{(-1)^k e^{-\frac{\gamma(k+1)}{\bar{\gamma}_i(1+(1-\rho_i^2)k)}}}{\bar{\gamma}_i(1+(1-\rho_i^2)k)} \quad (10)$$

$$\begin{aligned} f_{\gamma^*_{OSTBC}}(\gamma) &= 4\sum_{i=1}^{K}\sum_{k=0}^{K-1}\binom{K-1}{k}(-1)^k\sum_{n=0}^{k}\binom{k}{n}n!\gamma \\ &\quad \times \frac{(1-\rho_i^2)^n e^{-\frac{2\gamma(k+1)}{\bar{\gamma}_i(1+k(1-\rho_i^2))}} L_n^1\left(\frac{-2\gamma\rho_i^2}{\bar{\gamma}_i(1-\rho_i^2)\left(1+(1-\rho_i^2)k\right)}\right)}{\bar{\gamma}_i^2(1+(1-\rho_i^2)k)^{n+2}} \end{aligned} \quad (11)$$

where $L_n^\alpha(x)$ is the Laguerre polynomial [8, Eq. 8.970.1].

IV. System Capacity

Going one step beyond, in this section we derive the analytical expressions of the system capacity in the presence of delays in the feedback channel, which is given by

$$C = \int_0^\infty \log_2(1+\gamma) f_{\gamma^*}(\gamma)d\gamma \quad (12)$$

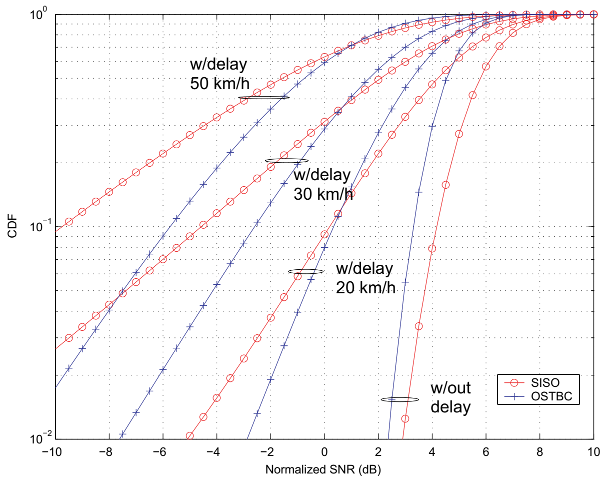

Fig. 1. CDF of the normalized post-scheduling SNR, $\frac{\gamma}{\bar{\gamma}}$, for the different transmission schemes and different MS speed (K=30 users).

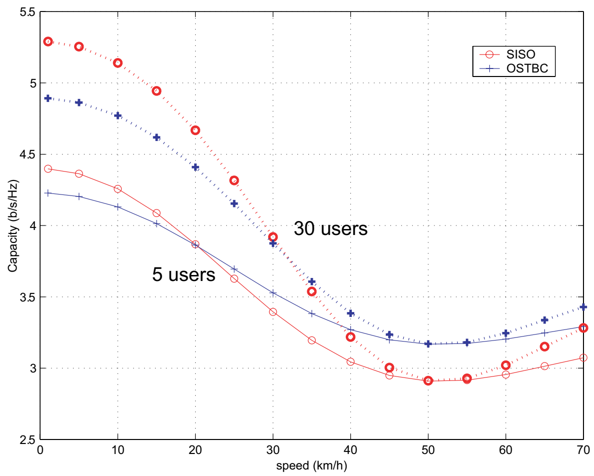

Fig. 2. Average system capacity vs. MS speed for the different transmission schemes ($K = 5$ and 30 users, $\bar{\gamma} = 10$ dB).

where the average is taken over channel (SNR) realizations and the corresponding pdf expressions can be found in Eqs. (10) and (11) above. For the SISO approach, we have that this integral can be solved with the help of [8, Eq. 4.331.2] and written in closed form as:

$$C_{SISO} = -\sum_{i=1}^{K} log_2 e \sum_{k=0}^{K-1} \binom{K-1}{k} \frac{(-1)^k}{k+1} e^{\frac{k+1}{\bar{\gamma}_i(1+(1-\rho_i^2)k)}} \times E_i\left(-\frac{k+1}{\bar{\gamma}_i(1+(1-\rho_i^2)k)}\right) \quad (13)$$

with $E_i(x)$ standing for the exponential integral function ($E_i(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt$, for $x < 0$). Analogously, for the OSTBC case we have that (12) can be analytically solved by resorting to [9, Eq. 78]. By doing so, the average system capacity can be expressed in terms of the complementary incomplete gamma function ($\Gamma_c(n, x) = \int_x^{\infty} e^{-t} t^{n-1} dt$) as:

$$C_{OSTBC} = 4\sum_{i=1}^{K} \log_2 e \sum_{k=0}^{K-1} \binom{K-1}{k} (-1)^k \sum_{n=0}^{k} \binom{k}{n} n! \times \sum_{m=0}^{n} \binom{n+1}{n-m} \rho_i^{2m} (1-\rho_i^2)^{n-m} (m+1) e^{\frac{2(k+1)}{\bar{\gamma}_i(1+k(1-\rho_i^2))}} \times \sum_{l=1}^{m+2} \frac{2^{m-l}\Gamma_c(l-m-2, \frac{2(k+1)}{\bar{\gamma}_i(1+(1-\rho_i^2)k)})}{\bar{\gamma}_i^{m+2-l}(k+1)^l(1+(1-\rho_i^2)k)^{m+n-l+2}} \quad (14)$$

V. Numerical Results and Discussion

In this section, we are interested in assessing spatial vs. multi-user trade-offs in situations where the partial CSI available at the BS is subject to delays. Due to space constraints, we will restrict ourselves to only show results corresponding to an homogenous case, i.e., $\bar{\gamma}_i = \bar{\gamma}$. First, the cumulative densities functions derived in Section III are plotted in Fig. 1 as a function of MS speed. As for the CSI delay, we adopt the parameters used in [5] for a High Speed Downlink Packet Access scenario where the authors justify that scheduling decisions can be made every 2 ms with a time delay of $T = 4$ ms. In the absence of delay, the SISO approach is far more effective than its OSTBC counterpart in shifting curves towards higher values of the post-scheduling SNR, which is clearly beneficial in terms of system capacity. For delayed feedback channels, though, the degradation experienced by the SISO scheme as the MS speed increases is larger than that of OSTBC. In other words, the single-antenna approach is less robust to channel uncertainty arising from CSI delays. For v=50 km/h, the relative ordering of both curves is reversed almost for the whole SNR range. As shown in Fig. 2, such behavior can be translated into system capacity measures. In particular, better results are obtained with OSTBC in scenarios with increasing MS speed. However, as the number of active users grows, the post-scheduling SNR peaks generated by the SISO configuration grow faster than those of OSTBC. As a result the SISO configuration performs better than OSTBC for higher values of MS speed (i.e., larger user sets partially compensate for SNR uncertainties). Finally, one can also observe that beyond 50 km/h curves are driven again towards higher values of the system capacity. This is because under the assumption of a Jakes' scattering model, the correlation depends on the zero-order Bessel function of the first kind, which is not a monotonically-decreasing function. In conclusion, OSTBC-based schemes are more appropriate for high mobility scenarios, in particular for a reduced number of users.

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