Closed-Form Approximation for the Steady-State Performance of Second-Order Kalman Filters

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***Abstract*—The Kalman filter is adopted in a myriad of applications for providing the minimum mean square error estimation of time-varying parameters in a simple and systematic manner. However, determining the Kalman filter performance is not so straightforward, particularly when process noise is present. In that case, one must often resort to numerical evaluations of the recursive Bayesian Cramér–Rao bound, or alternatively to implement the filter and assess the performance through Montecarlo simulations. This letter is intended to circumvent this limitation. It proposes a closed-form approximation for the steady-state performance of a Kalman filter based on a second-order dynamic model, while at the same time providing a novel closed-form upper bound for the convergence time. These two results are obtained by reformulating the Kalman filter in batch mode and analyzing the inner structure of the Bayesian information matrix. Simulation results are provided to illustrate the goodness of the proposed approach.**

***Index Terms*—Bayesian filtering, convergence time, Cramér–Rao bounds, Kalman filters, steady-state performance.**

I. Introduction

KALMAN filter-based techniques are used in a wide range of disciplines for the minimum mean square error (MMSE) estimation of the sample function of some random process [1]. They can be found in target tracking [2], speech processing [3], carrier tracking for wireless communications and navigation [4], [5], channel estimation [6], and market model estimation in financial applications [7], just to mention a few. From a practical point of view, it is often interesting to determine the best achievable estimation performance in order to assess whether the user requirements can ultimately be met or not. To this end, one must resort to the evaluation of the Bayesian Cramér–Rao bound (BCRB) or the so-called posterior CRB [8]. For the problem at hand, it has the form of a discrete-time Riccati equation (DARE) that can be evaluated recursively [9]. Unfortunately, a closed-form solution for the BCRB is difficult to obtain, particularly when it reaches a stationary value due to the presence of process noise. This often hampers the analysis and tuning of the Kalman filter in practice, thus forcing the designer to resort to numerical evaluations or extensive simulations.

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The problem of assessing the Kalman filter performance has received significant interest in the literature. However, it is often difficult to find simple and clear closed-form expressions for the BCRB that avoid resorting to numerical or recursive evaluations. For instance, many works address the solution of the DARE, but the proposed methods typically involve eigendecompositions [10], [11]. Closed-form solutions have been proposed in [12] and [13] for second-order models provided that accurate *a priori* information is available, which is often unrealistic in many applications. Other works provide simplified versions of the BCRB at the expense of restricting to the low- and high-SNR asymptotes of the BCRB [14], even though some numerical evaluations are still needed at some point.

This letter proposes an alternative approach to compute the BCRB of Kalman filters that is based on reformulating the problem in batch mode, thus leaving the estimates at the current time instant as a function of all past measurements in the data record. Assuming a linear Gaussian model and a diffuse initialization of the Kalman filter (i.e., noninformative prior), a parallelism can be made between the Kalman filter and the best linear unbiased estimator (BLUE). By exploiting this parallelism we are able to derive a closed-form approximation of the estimates provided by the Kalman filter and thus of the corresponding performance lower bound. Furthermore, we also analyze the steady-state behavior of the BCRB when process noise is present, which causes the BCRB to exhibit a floor effect [1]. For this phenomenon, we propose a closed-form expression to determine the convergence time of the filter. This becomes a valuable tool for tuning the Kalman filter in practice and, to our best, is a novel contribution of this letter.

To illustrate our approach, the scope of this letter is focused on second-order dynamic or kinematic models, which are representative of many applications in practice. In that sense, the contribution of this letter is twofold. First, a closed-form approximated upper bound for the convergence time of the corresponding Kalman filter is provided. Second, an approximation for the steady-state performance of the Kalman filter is presented. The proposed approach is generic and can thus be applied to models with either real or complex parameters. Although a second-order model is considered here, the proposed method could eventually be extended to higher order models, even though the derivation of closed-form expressions is not so mathematically tractable and remains out of the scope of the present letter.

II. Batch Formulation of the Kalman Filter

A. Kalman Filter State-Space Model

Let us assume a linear Kalman filter where the state transition equation is given by

$$\mathbf{x}(n+1) = \mathbf{F}\mathbf{x}(n) + \mathbf{G}v(n) \tag{1}$$

where, for the second-order case, $\mathbf{x}(n)$ is the two-dimensional state vector containing the sample functions of the processes we aim to estimate, and the propagation from $\mathbf{x}(n)$ toward $\mathbf{x}(n+1)$ is done through the (2×2) transition matrix $\mathbf{F}$. The term $v(n)$ is the so-called process noise, which accounts for the possible discrepancies between the Kalman state-space model and the actual input signal model, and is modeled as $v(n) \sim \mathcal{N}(0, \sigma_v^2)$ with $\mathrm{E}[v(i)v^*(j)] = 0$ for $i \neq j$. The process noise propagates to the rest of states through the constant (2×1) matrix $\mathbf{G}$, and the resulting noise is modeled as a zero-mean Gaussian process with covariance matrix $\mathbf{Q}$

$$\mathbf{Q} \doteq \mathrm{E}\left[|v(n)|^2 \mathbf{G}\mathbf{G}^H\right] = \sigma_v^2 \mathbf{G}\mathbf{G}^H. \tag{2}$$

To estimate the sample functions in $\mathbf{x}(n)$, we have the following scalar measurements $z(n)$ available

$$z(n) = \mathbf{H}\mathbf{x}(n) + w(n) \tag{3}$$

with $\mathbf{H}$ the (1×2) observation matrix and $w(n)$ the measurement noise corrupting the observations, which is modeled as $w(n) \sim \mathcal{N}(0, \sigma_w^2)$ with $\mathrm{E}[w(i)w^*(j)] = 0$ for $i \neq j$. The process and measurement noises are independent zero-mean Gaussian processes, and thus, $\mathrm{E}[v(i)w^*(j)] = \mathrm{E}[w(i)v^*(j)] = 0$ for all values of i and j. In many applications such as radar tracking, the process noise is small compared to the measurement noise and the system becomes mainly dominated by the latter [1]. In this letter, we will follow the same approach assuming that $\sigma_w^2 \gg \sigma_v^2$.

B. Kalman Filter Estimates and Recursive BCRB

For each of the incoming measurements, the Kalman filter provides the sequential MMSE estimate of $\mathbf{x}(n)$ as

$$\hat{\mathbf{x}}(n) = \hat{\mathbf{x}}(n|n-1) + \mathbf{K}(n)\left(z(n) - \hat{z}(n)\right) \tag{4}$$

where $\hat{\mathbf{x}}(n|n-1) = \mathbf{F}\hat{\mathbf{x}}(n-1)$ is the propagated state vector, $\mathbf{K}(n)$ is denoted as the Kalman gain, and $\hat{z}(n) = \mathbf{H}\hat{\mathbf{x}}(n|n-1)$ is the measurement prediction [15]. The estimation performance is given by the covariance matrix of the state estimate, $\boldsymbol{\Sigma}_{\mathbf{x}}(n) \doteq \mathrm{E}[(\hat{\mathbf{x}}(n) - \mathbf{x}(n))(\hat{\mathbf{x}}(n) - \mathbf{x}(n))^H]$, which is lower bounded by the BCRB:

$$\boldsymbol{\Sigma}_{\mathbf{x}}(n) \geq \mathbf{J}_{\mathrm{B}}^{-1}(n) \tag{5}$$

with $\mathbf{J}_{\mathrm{B}}(n)$ the Bayesian information matrix (BIM). Even though there exists no closed-form expression of the BCRB for the linear Gaussian model under consideration, the BIM can be computed recursively as [1]

$$\mathbf{J}_{\mathrm{B}}(n) = \left[\mathbf{Q} + \mathbf{F}\mathbf{J}_{\mathrm{B}}^{-1}(n-1)\mathbf{F}^H\right]^{-1} + \sigma_w^{-2}\mathbf{H}^H\mathbf{H} \tag{6}$$

which simplifies the computation of the BCRB. However, this expression still lacks the insights that a truly closed-form expression can unveil.

C. Batch-Mode Estimates and Performance Lower Bound

Let us first start by stacking the measurements in (3) for all time instants up to n into the following $(n \times 1)$ vector

$$\mathbf{z}_n \doteq \left[\, z(1) \quad z(2) \quad \cdots \quad z(n) \,\right]^T. \tag{7}$$

We assume that $\mathbf{F}$ is nonsingular and therefore the problem is well posed for state estimation.[1] Then, (1) can be expressed in terms of $\mathbf{x}(n)$ as a function of $\mathbf{x}(n+1)$ as

$$\mathbf{x}(n) = \mathbf{F}^{-1}\left(\mathbf{x}(n+1) - \mathbf{G}v(n)\right) \tag{8}$$

which allows $\mathbf{x}(n-\tau)$ to be computed as a function of $\mathbf{x}(n)$ for $\tau = \{1, 2, \ldots, (n-1)\}$. In this way, the measurement model at any $(n-\tau)$ can be written as

$$z(n-\tau) = \mathbf{H}\mathbf{F}^{-\tau}\mathbf{x}(n) - \sum_{l=1}^{\tau} \mathbf{H}\mathbf{F}^{-(\tau-l+1)}\mathbf{G}v(n-l) + w(n-\tau) \tag{9}$$

thus leading to the following batch-mode signal model as a function of the Kalman state vector at time n, $\mathbf{x}(n)$. That is

$$\mathbf{z}_n = \mathbf{A}_n\mathbf{x}(n) + \mathbf{B}_n\mathbf{u}_n \tag{10}$$

where $\mathbf{A}_n$, $\mathbf{B}_n$, and $\mathbf{u}_n$ are $(n \times 2)$, $(n \times (2n-1))$, and $((2n-1) \times 1)$ matrices given by

$$\mathbf{A}_n \doteq \left[\, \mathbf{H}\mathbf{F}^{-(n-1)}; \quad \mathbf{H}\mathbf{F}^{-(n-2)}; \quad \cdots; \quad \mathbf{H} \,\right] \tag{11}$$

$$\mathbf{B}_n \doteq \left[\mathbf{I}_n \;\; \begin{bmatrix} -\mathbf{H}\mathbf{F}^{-1}\mathbf{G} & \cdots & -\mathbf{H}\mathbf{F}^{-(n-1)}\mathbf{G} \\ 0 & \cdots & -\mathbf{H}\mathbf{F}^{-(n-2)}\mathbf{G} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\mathbf{H}\mathbf{F}^{-1}\mathbf{G} \\ 0 & \cdots & 0 \end{bmatrix} \right] \tag{12}$$

$$\mathbf{u}_n \doteq \left[\, w(1)\; w(2) \cdots w(n)\; v(1)\; v(2) \cdots v(n-1) \,\right]^T \tag{13}$$

with $\mathbf{I}_n$ the $(n \times n)$ identity matrix. The Kalman filter is known to provide the linear MMSE (LMMSE) estimate of $\mathbf{x}(n)$. However, when considering a diffuse initialization of the Kalman filter, the problem in (10) can be reinterpreted as that of estimating a vector of unknown deterministic parameters $\mathbf{x}(n)$ from Gaussian measurements $\mathbf{z}_n \sim \mathcal{N}(\mathbf{A}_n\mathbf{x}(n), \boldsymbol{\Sigma}_{\mathbf{z}_n})$, with $(n \times n)$ covariance matrix

$$\boldsymbol{\Sigma}_{\mathbf{z}_n} \doteq \mathrm{E}\left[\mathbf{B}_n\mathbf{u}_n\mathbf{u}_n^H\mathbf{B}_n^H\right] = \mathbf{B}_n\boldsymbol{\Sigma}_{\mathbf{u}_n}\mathbf{B}_n^H \tag{14}$$

where $\boldsymbol{\Sigma}_{\mathbf{u}_n} \doteq \mathrm{diag}([\,\boldsymbol{\sigma}^2_{w,(1\times n)} \quad \boldsymbol{\sigma}^2_{v,(1\times(n-1))}\,])$ encompasses both the measurement and the process noise variances for all time instants. Given the linear nature of the problem, the optimal batch-mode estimator of the state vector $\mathbf{x}(n)$ is given by the BLUE [17]

$$\hat{\mathbf{x}}(n) = \left(\mathbf{A}_n^H\boldsymbol{\Sigma}_{\mathbf{z}_n}^{-1}\mathbf{A}_n\right)^{-1}\mathbf{A}_n^H\boldsymbol{\Sigma}_{\mathbf{z}_n}^{-1}\mathbf{z}_n \tag{15}$$

which is also the minimum variance unbiased (MVU) estimator, thus attaining the corresponding (2×2) Fisher information matrix (FIM)

$$\mathbf{J}(n) = \mathbf{A}_n^H\boldsymbol{\Sigma}_{\mathbf{z}_n}^{-1}\mathbf{A}_n. \tag{16}$$

In the absence of *a priori* information, both the Bayesian and the frequentist approaches coincide [17], thus leading to $\mathbf{J}_{\mathrm{B}}(n) = \mathbf{J}(n)$. Therefore, we will henceforth make use of (16) to lower bound the estimation performance of the Kalman filter, namely $\boldsymbol{\Sigma}_{\mathbf{x}}(n) \geq \mathbf{J}^{-1}(n)$.

III. Fisher Information Matrix

A. Inner Structure of the Fisher Information Matrix

In order to derive a closed-form expression for the FIM in (16) we first need to get some insight into the inner structure of $\boldsymbol{\Sigma}_{\mathbf{z}_n}$. To do so, we will exploit the result in (14) and the inner

[1] This makes the system *reachable*, so there exists a finite sequence of $v(n)$, $n = 0, 1, n'$ such that any initial state vector $\mathbf{x}(0)$ can be transferred to any final state $\mathbf{x}(n'+1)$ [16].

structure of both $\mathbf{B}_n$ and $\mathbf{\Sigma}_{\mathbf{u}_n}$. Matrix $\mathbf{\Sigma}_{\mathbf{z}_n}$ can be decomposed into two terms as follows

$$\mathbf{\Sigma}_{\mathbf{z}_n} = \sigma_v^2 \mathbf{M}_n + \sigma_w^2 \mathbf{I}_n \tag{17}$$

with $\mathbf{M}_n$ a nonnegative $(n \times n)$ symmetric matrix whose constituent elements are formed by cross-products of matrices $\mathbf{H}$, $\mathbf{F}$, and $\mathbf{G}$ according to (14). By further inspection, it is found that the main diagonal elements of $\mathbf{M}_n$ are given by

$$[\mathbf{M}_n]_{k,k} = \sum_{m=1}^{n-k} \left(\mathbf{H}\mathbf{F}^{-m}\mathbf{G}\right)^2. \tag{18}$$

The computation of the FIM in (16) requires the inverse matrix of $\mathbf{\Sigma}_{\mathbf{z}_n}$. It is interesting to compute $\mathbf{\Sigma}_{\mathbf{z}_n}^{-1}$ using a single expression that avoids the inverse operation as such. For this purpose, we take advantage of the knowledge of $\mathbf{M}_n$, and we propose the following approximation for $\mathbf{\Sigma}_{\mathbf{z}_n}$

$$\widetilde{\mathbf{\Sigma}}_{\mathbf{z}_n} \doteq \sigma_v^2 \widetilde{\mathbf{M}}_n + \sigma_w^2 \mathbf{I}_n \tag{19}$$

where $\widetilde{\mathbf{M}}_n \doteq \mathbf{d}_{\mathbf{M}_n} \mathbf{d}_{\mathbf{M}_n}^H$ is defined herein as an approximated version of $\mathbf{M}_n$ with $\mathbf{d}_{\mathbf{M}_n} \doteq \sqrt{\mathrm{diag}(\mathbf{M}_n)}$ the $(n \times 1)$ vector containing the diagonal elements of $\mathbf{M}_n$ in (18).

B. FIM for Second-Order Dynamic Models

In the remainder of this letter, we consider a time-varying magnitude whose discrete-time evolution follows a second-order dynamic model, which can be understood as a random walk with a random-walk drift. The following state-space model is adopted:

$$\begin{bmatrix} \theta(n) \\ \dot{\theta}(n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta(n-1) \\ \dot{\theta}(n-1) \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} v(n) \tag{20}$$

where, according to Section II-A, we define $\mathbf{x}(n) \doteq [\theta(n) \quad \dot{\theta}(n)]^T$, $\mathbf{F} \doteq \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, and $\mathbf{G} \doteq [1/2 \quad 1]^T$. The measurements for the problem at hand are given by

$$z(n) = \theta(n) + w(n) \tag{21}$$

and therefore by direct comparison with (3), $\mathbf{H} \doteq [1 \quad 0]$.

The diagonal elements of $\mathbf{M}_n$ in (18) depend on the product $\mathbf{H}\mathbf{F}^{-m}\mathbf{G}$. Substituting the matrices $\mathbf{H}$, $\mathbf{F}$, and $\mathbf{G}$ introduced above, it can be found that $\mathbf{H}\mathbf{F}^{-m}\mathbf{G} = \frac{1}{2} - m$. Then, substituting into (18), the diagonal elements of $\mathbf{M}_n$ and therefore the constituent elements of $\widetilde{\mathbf{M}}_n$ become

$$[\mathbf{d}_{\mathbf{M}_n}]_{n-k} = \sqrt{\frac{k}{12}(4k^2 - 1)}. \tag{22}$$

Model orders higher than two would lead to an extended version of (22), even though closed-form expressions are difficult to find. This seriously hinders the subsequent derivation of closed-form expressions for the FIM and the convergence time, thus leaving this approach out of the scope of this letter.

In Fig. 1, we evaluate the tightness of the approximation above choosing $\sigma_w^2 = 10^{-5}$ and $\sigma_v^2 = 10^{-8}$ consistently with the assumption that $\sigma_w^2 \gg \sigma_v^2$. Fig. 1 depicts the BCRB for $[\mathbf{J}_B^{-1}(n)]_{1,1}$ compared to $[\mathbf{J}^{-1}(n)]_{1,1}$ under the approximated version of $\mathbf{M}_n$ and also when neglecting $\mathbf{M}_n$ in (17) (i.e., zero process-noise). The approximation is seen to provide a tight match with the BCRB while the Kalman filter is in its transient stage and also at the starting region of the steady state. In contrast, when neglecting $\mathbf{M}_n$ we never reach such steady state. This observation supports the importance of $\mathbf{M}_n$ and its proposed approximation, which greatly simplifies the computation of both the Kalman convergence time and the performance bounds in steady state.

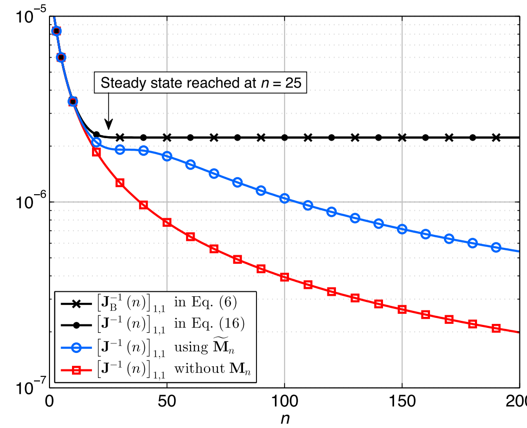

Fig. 1. Comparison between recursive BCRB for $[\mathbf{J}_B^{-1}(n)]_{1,1}$ and CRB for $[\mathbf{J}^{-1}(n)]_{1,1}$ with different versions of $\mathbf{M}_n$.

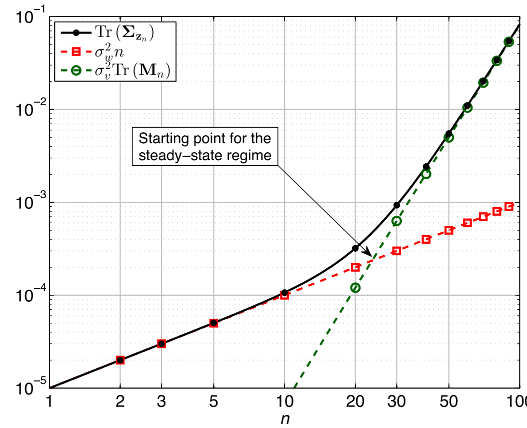

Fig. 2. Asymptotic evolution of $\mathrm{Tr}(\mathbf{\Sigma}_{\mathbf{z}_n})$ versus the constituent elements of $\mathbf{\Sigma}_{\mathbf{z}_n}$ in (17).

IV. Approximation of Kalman Filter Convergence Time and Steady-State Performance

A. Closed-Form Approximation for the Convergence Time

The proposed approach for determining the Kalman convergence time is based on observing the evolution of the trace of $\mathbf{\Sigma}_{\mathbf{z}_n}$ in (17) over n. For small n, it is dominated by the component of the measurement noise σ_w^2. When enlarging the data record, there is a change of trend at some point at which the component of the process noise σ_v^2 starts being dominant. This situation remains then for $n \to \infty$ (i.e., steady state). That point becomes thus the turning point that determines the transition toward the steady-state region. Therefore, the intersection point between both contributions is considered to be given when the Kalman filter reaches its steady state.

Fig. 2 illustrates the phenomenon above by comparing the trace of $\mathbf{\Sigma}_{\mathbf{z}_n}$ to the measurement and process noise contributions in (17) using the same values of σ_w^2 and σ_v^2 employed in Fig. 1. Both contributions intersect at the sample index $n = 25$, which coincides with the convergence time in Fig. 1.

In order to derive the Kalman convergence time, the decomposition of $\mathbf{\Sigma}_{\mathbf{z}_n}$ in (17) fortunately allows us to provide a simple and compact expression to work with. Using (22) for the diagonal elements of $\mathbf{M}_n$, the trace of $\mathbf{M}_n$ is given by

$$\mathrm{Tr}\left(\mathbf{M}_n\right) = \sum_{k=0}^{n-1} \frac{k}{12}\left(4k^2-1\right) = \frac{n}{24}\left(n-1\right)\left(2n^2-2n-1\right). \tag{23}$$

With this, the trace of $\boldsymbol{\Sigma}_{\mathbf{z}_n}$ becomes

$$\mathrm{Tr}\left(\boldsymbol{\Sigma}_{\mathbf{z}_n}\right) = \frac{\sigma_v^2}{24}\gamma(n) + \sigma_w^2 n \tag{24}$$

with $\gamma(n) \doteq n(n-1)(2n^2-2n-1)$, and therefore, the time at which the intersection point takes place, denoted herein as n_{c}, can be found as the solution to the equality $\frac{\sigma_v^2}{24}\gamma(n) = \sigma_w^2 n$. Substituting $\gamma(n)$ and after some manipulations, we find that the Kalman filter convergence time can be obtained as the real positive root of the following third-order polynomial:

$$f_{\mathrm{c}}(n) = 2n^3 - 4n^2 + n + \left(1 - 24\frac{\sigma_w^2}{\sigma_v^2}\right). \tag{25}$$

The polynomial in (25) has only one real root (i.e., unique real solution for n_{c}) when the discriminant is $\Delta(f_c) < 0$, which is guaranteed when $\sigma_w^2 \gg \sigma_v^2$. Under this assumption, and after some more mathematical manipulations, it is found that the convergence time can be approximated in closed-form by

$$n_{\mathrm{c}} \approx \frac{2}{3} + \left(12\frac{\sigma_w^2}{\sigma_v^2}\right)^{1/3} \doteq \tilde{n}_{\mathrm{c}}. \tag{26}$$

The lack of *a priori* knowledge due to the diffuse initialization leads the convergence time to be maximum, since the filter has initially no clue (i.e., maximum initial uncertainty) about the parameters to be estimated. The result above can therefore be understood as an upper bound on the convergence time.

B. Closed-Form Approximation for Steady-State Performance

By inspection of (16) the entries of the FIM are given by

$$\left[\mathbf{J}(n)\right]_{1,1} = \mathbf{1}^T\boldsymbol{\Sigma}_{\mathbf{z}_n}^{-1}\mathbf{1} \tag{27}$$

$$\left[\mathbf{J}(n)\right]_{1,2} = -\boldsymbol{\eta}_n^T\boldsymbol{\Sigma}_{\mathbf{z}_n}^{-1}\mathbf{1} \tag{28}$$

$$\left[\mathbf{J}(n)\right]_{2,2} = \boldsymbol{\eta}_n^T\boldsymbol{\Sigma}_{\mathbf{z}_n}^{-1}\boldsymbol{\eta}_n \tag{29}$$

where $[\mathbf{J}(n)]_{1,2} = [\mathbf{J}(n)]_{2,1}$. The expressions above manifest that the FIM in (16) involves the computation of $\boldsymbol{\Sigma}_{\mathbf{z}_n}^{-1}$. Thanks to the decomposition in (17) and the approximation of $\mathbf{M}_n$ in (19), we can apply the matrix inversion lemma on the inverse of (19) to obtain a closed-form approximation for $\boldsymbol{\Sigma}_{\mathbf{z}_n}^{-1}$ as

$$\widetilde{\boldsymbol{\Sigma}}_{\mathbf{z}_n}^{-1} = \frac{1}{\sigma_w^2}\left(\mathbf{I}_n - \frac{6\sigma_v^2}{24\sigma_w^2 + \gamma(n)\sigma_v^2}\mathbf{d}_{\mathbf{M}_n}\mathbf{d}_{\mathbf{M}_n}^T\right). \tag{30}$$

Replacing $\boldsymbol{\Sigma}_{\mathbf{z}_n}^{-1}$ in (27)–(29) with the approximation in (30), we obtain the following closed-form approximation of the FIM:

$$\widetilde{\mathbf{J}}^{-1}(n) = \frac{1}{\alpha(n)}\begin{bmatrix} \boldsymbol{\eta}_n^T\widetilde{\boldsymbol{\Sigma}}_{\mathbf{z}_n}^{-1}\boldsymbol{\eta}_n & \boldsymbol{\eta}_n^T\widetilde{\boldsymbol{\Sigma}}_{\mathbf{z}_n}^{-1}\mathbf{1} \\ \boldsymbol{\eta}_n^T\widetilde{\boldsymbol{\Sigma}}_{\mathbf{z}_n}^{-1}\mathbf{1} & \mathbf{1}^T\widetilde{\boldsymbol{\Sigma}}_{\mathbf{z}_n}^{-1}\mathbf{1} \end{bmatrix} \tag{31}$$

where $\alpha(n) \doteq \mathbf{1}^T\widetilde{\boldsymbol{\Sigma}}_{\mathbf{z}_n}^{-1}\mathbf{1}\boldsymbol{\eta}_n^T\widetilde{\boldsymbol{\Sigma}}_{\mathbf{z}_n}^{-1}\boldsymbol{\eta}_n - \boldsymbol{\eta}_n^T\widetilde{\boldsymbol{\Sigma}}_{\mathbf{z}_n}^{-1}\mathbf{1}\boldsymbol{\eta}_n^T\widetilde{\boldsymbol{\Sigma}}_{\mathbf{z}_n}^{-1}\mathbf{1}$ and $\boldsymbol{\eta}_n \doteq [(n-1), (n-2), \ldots, 0]^T$. However, as shown previously in Fig. 1, this approximation can be applied up to the beginning of the steady state, $n = n_{\mathrm{c}}$, and loses its validity

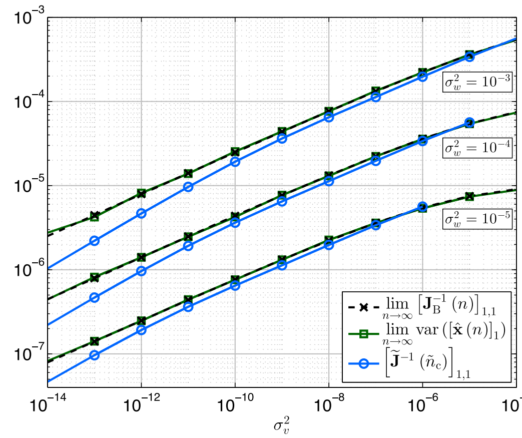

Fig. 3. Comparison between steady-state estimation performance and proposed approximation for $[\mathbf{J}^{-1}(n)]_{1,1}$ in (32).

when $n \to \infty$. Therefore, we approximate the performance lower bounds for the parameters of interest in steady state as

$$\lim_{n\to\infty}\left[\mathbf{J}_{\mathrm{B}}^{-1}(n)\right]_{1,1} \approx \left[\widetilde{\mathbf{J}}^{-1}(n_{\mathrm{c}})\right]_{1,1} = \frac{\boldsymbol{\eta}_{n_{\mathrm{c}}}^T\widetilde{\boldsymbol{\Sigma}}_{\mathbf{z}_{n_{\mathrm{c}}}}^{-1}\boldsymbol{\eta}_{n_{\mathrm{c}}}}{\alpha(n_{\mathrm{c}})} \tag{32}$$

$$\lim_{n\to\infty}\left[\mathbf{J}_{\mathrm{B}}^{-1}(n)\right]_{2,2} \approx \left[\widetilde{\mathbf{J}}^{-1}(n_{\mathrm{c}})\right]_{2,2} = \frac{\mathbf{1}^T\widetilde{\boldsymbol{\Sigma}}_{\mathbf{z}_{n_{\mathrm{c}}}}^{-1}\mathbf{1}}{\alpha(n_{\mathrm{c}})}. \tag{33}$$

To illustrate the applicability of our approach in a broader domain than Figs. 1 and 2, simulation results are provided in Fig. 3 for a wider range of values of σ_w^2 and σ_v^2, under the condition $\sigma_w^2 \gg \sigma_v^2$. The experimental performance of the Kalman filter given by the variance of $[\hat{\mathbf{x}}(n)]_1$ in steady state is compared to the approximation of $[\mathbf{J}^{-1}(n)]_{1,1}$ in (32) using $\tilde{n}_{\mathrm{c}}$ in (26). A tight match between the exact result and the proposed approach can be drawn in the central operating region. Some discrepancies appear when $\sigma_v^2 \to 0$, but this is consistent with the fact that the proposed method is applicable on the performance floor of the Kalman filter, which appears under the steady-state regime whenever $\sigma_v^2 \neq 0$. It is for this reason that mismatches tend to attenuate when σ_v^2 departs from zero, as observed in the figure. With this, we can confirm the effectiveness of the proposed approach for low-medium (i.e., $\sigma_v^2 \sim 10^{-10}$) to high σ_v^2, up to the point where the condition $\sigma_w^2 \gg \sigma_v^2$ no longer holds.

V. Conclusion

In this letter, two major results have been presented. First, a novel closed-form approximated upper bound for the convergence time of a second-order Kalman filter. Second, a closed-form approximated lower bound for the estimation performance in steady state. Both results have been obtained by reformulating the Kalman problem in batch mode and analyzing the behavior of the constituent elements of the Fisher information matrix. These expressions allow characterizing the Kalman filter explicitly as a function of the parameters of the problem, an information that remains somehow hidden in the recursive BCRB. Therefore, the results provided herein become a valuable tool for the practical tuning and performance evaluation of Kalman filter-based applications.

References

[1] H. L. V. Trees and K. L. Bell, *Bayesian Bounds for Parameter Estimation and Nonlinear Filtering/Tracking*. New York, NY, USA: Wiley, 2007.

[2] C. R. Rao, C. R. Sastry, and B. Zhou, “Tracking the direction of arrival of multiple moving targets,” *IEEE Trans. Signal Process.*, vol. 42, no. 5, pp. 1133–1144, May 1994.

[3] M. Gabrea, E. Grivel, and M. Najun, “A single microphone Kalman filter-based noise canceller,” *IEEE Signal Process. Lett.*, vol. 6, no. 3, pp. 55–57, Mar. 1999.

[4] A. Patapoutian, “On phase-locked loops and Kalman filters,” *IEEE Trans. Commun.*, vol. 47, no. 5, pp. 670–672, May 1999.

[5] P. Y. C. Hwang and R. Grover Brown, “GPS navigation: combining pseudorange with continuous carrier phase using a Kalman filter,” *ION Navig.*, vol. 37, pp. 185–190, Summer 1990.

[6] D. N. Godard, “Channel equalization using a Kalman filter for fast data transmission,” *IBM J. Res. Devlop.*, vol. 18, pp. 267–273, May 1974.

[7] C. Wells, *The Kalman Filter in Finance*. Norwell, MA, USA: Kluwer, 1996.

[8] H. L. V. Trees, *Detection, Estimation and Modulation Theory*. New York, NY, USA: Wiley, 1968, vol. 1.

[9] P. Tichavský, C. H. Muravchik, and A. Nehorai, “Posterior Cramér-Rao bounds for discrete-time nonlinear filtering,” *IEEE Trans. Signal Process.*, vol. 46, no. 5, pp. 1386–1396, May 1998.

[10] D. Vaughan, “A nonrecursive algebraic solution for the discrete Riccati equation,” *IEEE Trans. Autom. Control*, vol. 15, no. 5, pp. 597–599, Oct 1970.

[11] F. M. Ham and R. G. Brown, “Observability, eigenvalues, and Kalman filtering,” *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-19, no. 2, pp. 269–273, Mar. 1983.

[12] Y. Qian, X. Cui, M. Ku, and Z. Feng, “Steady-state performance of Kalman filter for DPLL,” *Tsinghua Sci. Technol.*, vol. 14, no. 4, pp. 470–473, 2009.

[13] L. Ros and E. P. Simon, “Second-order modeling for Rayleigh flat fading channel estimation with Kalman filter,” in *Proc. Int. Conf. Digital Signal Process.*, 2011, pp. 1–6.

[14] S. Bay, C. Herzet, J. M. Brossier, J. P. Barbot, and B. Geller, “Analytic and asymptotic analysis of Bayesian Cramér-Rao bound for dynamical phase offset estimation,” *IEEE Trans. Signal Process.*, vol. 56, no. 1, pp. 61–70, Jan. 2008.

[15] D. Simon, *Optimal State Estimation: Kalman, H Infinity, and Nonlinear Approaches*. New York, NY, USA: Wiley, 2006.

[16] E. Hendricks, O. Jannerup, and P. Haase, *Linear Systems Control. Deterministic and Stochastic Methods*. New York, NY, USA: Springer, 2008.

[17] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*. Upper Saddle River, NJ, USA: Prentice-Hall, 1993.