Closed-Form Approximation for the Convergence Time of pth-Order Kalman Filters

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***Abstract*—The Kalman filter is used in a myriad of applications for estimating a set of time-varying parameters in the minimum mean square error sense. When designing the filter, one of the points of most relevant interest is predicting its estimation performance. However, it is often difficult to find expressions in closed form that allow obtaining this information in a straightforward manner. As a consequence, the designer usually requires resorting to the numerical evaluation of the Bayesian Cramér–Rao bound or the implementation and assessment of the filter through Monte Carlo simulations. In this regard, this letter intends to contribute with a novel closed-form upper bound for the convergence time of a Kalman filter. To this end, the Kalman filtering problem is reformulated in batch mode, and the corresponding Fisher information matrix is analyzed. The contribution presented herein is based on a generic dynamic model and is not restricted to any specific order, in contrast to existing contributions. Simulation results are provided to illustrate the goodness of the proposed approach.**

***Index Terms*—Bayesian filtering, convergence time, Cramér–Rao bounds, Kalman filters, steady state.**

I. Introduction

KALMAN-filter-based techniques are used in plenty of disciplines to provide the minimum mean square error estimation of a random process [1]. In time-invariant statistical models, the so-called Bayesian Cramér–Rao bound (BCRB), or posterior CRB [2], is a tool of great practical interest to determine the best achievable performance of these techniques and evaluate the fulfillment of the user requirements. For the problem at hand, it has the form of a discrete-time Riccati equation that can be solved recursively [3]. Unfortunately, a closed-form expression for unveiling the insights into the Kalman filter performance becomes nontrivial to obtain, thus forcing the designer to evaluate the BCRB by means of numerical evaluations or extensive simulations.

A number of works aiming at circumventing this limitation can be found in the literature. However, many of them rely on assumptions that are often unrealistic in practice. This is the case of [4] and [5], where the results are conditioned upon the availability of an accurate prior. Other works provide simplified versions of the BCRB, yet still require some numerical evaluations [6] or eigendecompositions [7], [8]. One last example is the adoption of low-order model restrictions to simplify the complexity of the problem and provide results that are easily tractable [9]. Therefore, deriving expressions in closed form that facilitate the analysis and tuning of the Kalman filter in practice still remains a difficult task.

In [10], we analyzed the steady-state behavior of the BCRB, which exhibits a floor effect in the presence of process noise [1]. One of our main contributions was the derivation of a closed-form expression to determine the convergence time of a second-order Kalman filter before entering the steady state. This was carried out assuming a linear Gaussian model for the input measurements and a diffuse initialization of the filter (i.e., noninformative prior), thus allowing the problem to be reformulated in batch mode and be linked with the best linear unbiased estimator (BLUE). We focused on a magnitude that evolved following a so-called dynamic or kinematic model [11], sometimes referred to as Newtonian model [12], which is representative of many real applications. Some examples are target tracking [13] and carrier phase and frequency estimation in wireless communications and navigation [14], [15]. Notwithstanding, the main limitation of [10] was the restriction of the work to second-order models, leaving higher order models out of the scope of the presentation, since the derivations resulted in quite a rather cumbersome formulation.

The objective of this letter is to break this barrier and extend the performance analysis to higher order models. In particular, we derive a closed-form upper bound for the convergence time that can be applied to Kalman filters of any nonrestricted order. The proposed approach makes no assumption on the nature of the input signal beyond the ones mentioned above, thus being a rather generic approach applicable to models with either real or complex parameters. The outcome of this work becomes a valuable tool for the practical tuning of the Kalman filter and, to our best, is a novel contribution to the literature.

II. Batch Formulation of the Kalman Filter

A. *Kalman Filter State-Space and Observation Models*

Let us assume a linear pth-order Kalman filter with the following discrete-time state transition equation:

$$\mathbf{x}_p(n+1) = \mathbf{F}_p \mathbf{x}_p(n) + \mathbf{G}_p v(n) \tag{1}$$

where $\mathbf{x}_p(n)$ is the p-dimensional state vector containing the parameters of interest at time sample n, which propagates toward $(n+1)$ through the nonsingular $(p \times p)$ transition matrix $\mathbf{F}_p$. The term $v(n)$ stands for the process noise aiming at covering the possible discrepancies between the Kalman state-

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space model and the actual input signal. It is usually modeled as $v(n) \sim \mathcal{N}\left(0, \sigma_v^2\right)$ with $\mathrm{E}\left[v(i)v^*(j)\right] = 0$ for $i \neq j$. Its weight onto the different Kalman states is adjusted through the linear $(p \times 1)$ transformation given by $\mathbf{G}_p$, thus resulting in a zero-mean Gaussian process with covariance matrix $\mathbf{Q}_p \doteq \mathrm{E}\left[|v(n)|^2 \mathbf{G}_p \mathbf{G}_p^H\right] = \sigma_v^2 \mathbf{G}_p \mathbf{G}_p^H$.

The Kalman filter aims at providing estimates of $\mathbf{x}_p(n)$ based on the available scalar noisy measurements modeled as

$$z(n) = \mathbf{H}_p \mathbf{x}_p(n) + w(n) \tag{2}$$

where $\mathbf{H}_p$ is the $(1 \times p)$ observation matrix and $w(n)$ is the measurement noise corrupting the input observations. It is usually modeled as $w(n) \sim \mathcal{N}\left(0, \sigma_w^2\right)$ with $\mathrm{E}\left[w(i)w^*(j)\right] = 0$ for $i \neq j$. The process and measurement noises are independent zero-mean Gaussian disturbances, and thus, $\mathrm{E}\left[v(i)w^*(j)\right] = \mathrm{E}\left[w(i)v^*(j)\right] = 0$ for all i and j.

B. Batch-Mode Formulation

By conveniently combining (1) and (2), we obtained [10, eq. (9)] that allowed us to stack all the measurements $z(n)$ in the data record up to time sample n into a vector defined as

$$\mathbf{z}_n \doteq \left[\, z(1)\ z(2)\ \cdots\ z(n)\,\right]^T. \tag{3}$$

This facilitated the formulation of the Kalman filter in batch mode as in [10, eq. (10)]

$$\mathbf{z}_n = \mathbf{A}_{n,p} \mathbf{x}_p(n) + \mathbf{B}_n \mathbf{u}_n \tag{4}$$

where $\mathbf{A}_{n,p}$, $\mathbf{B}_n$, and $\mathbf{u}_n$ are $(n \times p)$, $(n \times (2n-1))$, and $((2n-1) \times 1)$ matrices given by

$$\mathbf{A}_{n,p} \doteq \left[\mathbf{H}_p \mathbf{F}_p^{-(n-1)};\ \mathbf{H}_p \mathbf{F}_p^{-(n-2)};\ \cdots;\ \mathbf{H}_p\right] \tag{5}$$

$$\mathbf{B}_n \doteq \left[\mathbf{I}_n, \begin{bmatrix} -\mathbf{H}_p \mathbf{F}_p^{-1} \mathbf{G}_p & \cdots & -\mathbf{H}_p \mathbf{F}_p^{-(n-1)} \mathbf{G}_p \\ 0 & \cdots & -\mathbf{H}_p \mathbf{F}_p^{-(n-2)} \mathbf{G}_p \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\mathbf{H}_p \mathbf{F}_p^{-1} \mathbf{G}_p \\ 0 & \cdots & 0 \end{bmatrix}\right] \tag{6}$$

$$\mathbf{u}_n \doteq \left[\, w(1)\ w(2)\ \cdots\ w(n)\ v(1)\ v(2)\ \cdots\ v(n-1)\,\right]^T \tag{7}$$

with $\mathbf{I}_n$ the $(n \times n)$ identity matrix.

The estimation performance of the Kalman filter is lower bounded by the BCRB given by the inverse of the Bayesian information matrix, denoted as $\mathbf{J}_{\mathrm{B}}^{-1}(n)$. Notwithstanding, when considering a noninformative prior, a parallelism can be made between the Kalman filter and the optimal batch-mode estimator of $\mathbf{x}_p(n)$, namely the BLUE [16]. Interestingly, the latter is the minimum variance unbiased estimator for the linear Gaussian problem at hand, thus attaining the so-called Fisher information matrix (FIM) (see [10, eq. (16)])

$$\mathbf{J}^{-1}(n) = \left(\mathbf{A}_{n,p}^H \boldsymbol{\Sigma}_{\mathbf{z}_n}^{-1} \mathbf{A}_{n,p}\right)^{-1} \tag{8}$$

where $\boldsymbol{\Sigma}_{\mathbf{z}_n}$ is the $(n \times n)$ covariance matrix of the Gaussian measurements in (4), with $\mathbf{z}_n \sim \mathcal{N}\left(\mathbf{A}_{n,p}\mathbf{x}_p(n), \boldsymbol{\Sigma}_{\mathbf{z}_n}\right)$ and

$$\boldsymbol{\Sigma}_{\mathbf{z}_n} \doteq \mathrm{E}\left[\mathbf{B}_n \mathbf{u}_n \mathbf{u}_n^H \mathbf{B}_n^H\right] = \mathbf{B}_n \boldsymbol{\Sigma}_{\mathbf{u}_n} \mathbf{B}_n^H \tag{9}$$

with $\boldsymbol{\Sigma}_{\mathbf{u}_n} \doteq \mathrm{diag}\left(\left[\boldsymbol{\sigma}_{w,(1\times n)}^2, \boldsymbol{\sigma}_{v,(1\times(n-1))}^2\right]\right)$.

In the absence of *a priori* information, both the Bayesian and the frequentist approaches coincide [16], thus leading to $\mathbf{J}_{\mathrm{B}}(n) = \mathbf{J}(n)$. Therefore, we can make use of (8) to extract the information about the convergence time of the Kalman filter when it operates under a diffuse initialization.

III. Fisher Information Matrix

A. Inner Structure of the FIM

In order to determine the Kalman filter convergence time using the FIM in (8), we need to particularly get some insight into matrix $\boldsymbol{\Sigma}_{\mathbf{z}_n}$ in (9). By exploiting the knowledge of matrices $\mathbf{B}_n$ and $\boldsymbol{\Sigma}_{\mathbf{u}_n}$, the inner structure of $\boldsymbol{\Sigma}_{\mathbf{z}_n}$ for any p is found to be formed by two separate terms

$$\boldsymbol{\Sigma}_{\mathbf{z}_n} = \sigma_v^2 \mathbf{M}_n + \sigma_w^2 \mathbf{I}_n \tag{10}$$

with $\mathbf{M}_n$ a nonnegative $(n \times n)$ symmetric matrix whose constituent elements are formed by cross products of $\mathbf{H}_p$, $\mathbf{F}_p$, and $\mathbf{G}_p$. By further inspection, it is found that, in particular, the diagonal elements of $\mathbf{M}_n$ are driven by [10, eq. (18)]

$$[\mathbf{M}_n]_{k,k} = \sum_{m=1}^{n-k} \left(\mathbf{H}_p \mathbf{F}_p^{-m} \mathbf{G}_p\right)^2 \tag{11}$$

with $k \doteq [1..n]$. The knowledge of (11) becomes of great utility, since it is the information from which we can derive a closed-form expression for the Kalman convergence time, as will be shown in Section IV. The way forward requires thus the definition of a signal model that can be encompassed by matrices $\mathbf{H}_p$, $\mathbf{F}_p$, and $\mathbf{G}_p$ in order to keep working with (11).

B. Formulation of the pth-Order Kinematic Model

In the sequel, we consider a time-varying magnitude whose discrete-time evolution adopts the following generic pth-order dynamic or kinematic model [11, Ch. 6]:

$$\theta(n) = \theta(n-1) + \sum_{k=1}^{p} \frac{\theta^{(k)}(n-1)}{k!} \tag{12}$$

where the superindex (k) refers to the k th-order derivative of $\theta(n)$, which is considered to evolve as a kinematic model as well, but with order $(p-k)$

$$\theta^{(k)}(n) = \theta^{(k)}(n-1) + \sum_{l=1}^{p-k} \frac{\theta^{(k+l)}(n-1)}{l!}. \tag{13}$$

The models in (12) and (13) can be jointly rewritten as

$$\begin{bmatrix} \theta(n) \\ \theta^{(1)}(n) \\ \theta^{(2)}(n) \\ \vdots \\ \theta^{(p-1)}(n) \end{bmatrix} = \mathbf{F}_p \begin{bmatrix} \theta(n-1) \\ \theta^{(1)}(n-1) \\ \theta^{(2)}(n-1) \\ \vdots \\ \theta^{(p-1)}(n-1) \end{bmatrix} + \mathbf{G}_p v(n) \tag{14}$$

where $\mathbf{x}_p(n) \doteq \left[\,\theta(n)\ \theta^{(1)}(n)\ \theta^{(2)}(n)\ \cdots\ \theta^{(p-1)}(n)\,\right]^T$ defines the Kalman state vector, and $v(n) \doteq \theta^{(p)}(n)$. $\mathbf{F}_p$ and $\mathbf{G}_p$ are the $(p \times p)$ and $(p \times 1)$ transition and process noise propagation matrices defined as

$$\mathbf{F}_p \doteq \begin{bmatrix} 1 & \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{(p-1)!} \\ 0 & 1 & \frac{1}{1!} & \cdots & \frac{1}{(p-2)!} \\ 0 & 0 & 1 & \cdots & \frac{1}{(p-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{15}$$

$$\mathbf{G}_p \doteq \left[\frac{1}{p!}\ \ \frac{1}{(p-1)!}\ \ \frac{1}{(p-2)!}\ \ \cdots\ \ \frac{1}{1!}\right]^T. \tag{16}$$

On the other side, the measurements for the problem at hand are given by $z(n) = \theta(n) + w(n)$, and based on (2), the $(1 \times p)$ measurement the matrix therefore becomes

$$\mathbf{H}_p \doteq [\,1\;0\;0\cdots\;0\,]. \tag{17}$$

IV. Upper Bound for Kalman Filter Convergence Time

A. Determination of Kalman Filter Convergence Time

Recalling [10, Sec. IV], the determination of the Kalman convergence time is based on observing the evolution of the trace of $\boldsymbol{\Sigma}_{\mathbf{z}_n}$ in (10). This was graphed in [10, Fig. 2], which also included the evolution of the two separate components given by $\sigma_v^2 \operatorname{Tr}(\mathbf{M}_n)$ and $\sigma_w^2 n$. For small n, the trace of $\boldsymbol{\Sigma}_{\mathbf{z}_n}$ is dominated by the contribution of the measurement noise σ_w^2 filtered by the Kalman. At a given time instant, a change of state is produced, in which $\operatorname{Tr}(\boldsymbol{\Sigma}_{\mathbf{z}_n})$ becomes dominated by the contribution of the process noise, a situation that remains then for $n \to \infty$. This is the point at which the BCRB becomes lower saturated by nonzero process noise [1]. Therefore, the intersection point of both components is considered to determine the time instant at which the Kalman enters the steady state. The latter can thus be found as the solution for n to the equality

$$\sigma_v^2 \operatorname{Tr}(\mathbf{M}_n) = \sigma_w^2 n. \tag{18}$$

The problem of solving (18) requires determining the trace of $\mathbf{M}_n$. To this end, we fortunately have expression (11) to work with. From this, the trace of $\mathbf{M}_n$ can be computed as

$$\operatorname{Tr}(\mathbf{M}_n) = \sum_{k=1}^{n} [\mathbf{M}_n]_{k,k} = \sum_{k=1}^{n} \sum_{m=1}^{n-k} \left(\mathbf{H}_p \mathbf{F}_p^{-m} \mathbf{G}_p\right)^2. \tag{19}$$

Substituting the matrices $\mathbf{H}_p$, $\mathbf{F}_p$, and $\mathbf{G}_p$ defined in (15)–(17) and evaluating for different model orders p, it is found that (19) can be rewritten as a polynomial with degree $2p$

$$\operatorname{Tr}(\mathbf{M}_n) = \sum_{q=1}^{2p} \beta'_q n^q \tag{20}$$

with $\{\beta'_q\}_{q=1}^{2p}$ a set of coefficients whose absolute value is found to be less than one. Substituting (20) into (18), the latter can be rewritten as

$$\sigma_v^2 \sum_{q=1}^{2p} \beta'_q n^q = \sigma_w^2 n. \tag{21}$$

In the equality in (21), both sides can be compensated by a factor of n. By doing so, we find that the convergence time of a pth-order Kalman filter, denoted henceforth as $n_{\mathrm{c}}^{(p)}$, can be obtained as the real positive root of the following polynomial with degree $(2p-1)$:

$$f_{\mathrm{c}}(n,p) = \beta_0 + \sum_{q=1}^{2p-1} \beta_q n^q \tag{22}$$

with $\beta_q \doteq \beta'_{q+1}$, and β_0 is the independent term defined as

$$\beta_0 \doteq \beta'_1 - \frac{\sigma_w^2}{\sigma_v^2}. \tag{23}$$

Deriving a closed-form solution for the real root of (22) may become a rather cumbersome mathematical problem when models with sufficiently high order are considered. For this reason, in Section IV-B, we formulate an approximation for $n_{\mathrm{c}}^{(p)}$ that considerably reduces the complexity of the problem.

B. Closed-Form Approximation for the Convergence Time

In order to derive a closed-form approximation for $n_{\mathrm{c}}^{(p)}$, we will henceforth assume that $\sigma_w^2 \gg \sigma_v^2$. This assumption is widely considered in many practical applications such as the radar/sonar tracking problem [1] and the one in which the potential of the Kalman filter can mostly be exploited. In this situation, the Kalman estimates the parameters of interest by filtering the input measurement noise, down until reaching the floor saturation imposed by nonzero process noise. Such a floor effect tends to disappear when $\sigma_v^2 \to 0$, causing the BCRB to decrease without bound and never attain a steady state, leading to $n_{\mathrm{c}}^{(p)} \to \infty$. On the contrary, the floor becomes more stringent when gradually increasing σ_v^2, causing the convergence time to reduce. When σ_v^2 is such that the condition $\sigma_w^2 \gg \sigma_v^2$ is not fulfilled, the Kalman is left with no margin to filter the measurement noise. Consequently, the process noise becomes rapidly the dominating effect in this situation, thus leading the Kalman filter to saturate at the very first iteration, that is, $n_{\mathrm{c}}^{(p)} \to 1$.

Therefore, as a general rule, we have that $1 \leq n_{\mathrm{c}}^{(p)} < \infty$. In this way, we can derive a closed-form approximation of $n_{\mathrm{c}}^{(p)}$ through asymptotic analysis. If we consider σ_v^2 small compared to σ_w^2, we can claim from the reasoning above that $n_{\mathrm{c}}^{(p)} > 1$. Being this raised up to $(2p-1)$ in (22), the polynomial becomes rapidly dominated by the highest order term, namely,

$$f_{\mathrm{c}}(n,p) \approx \beta_{2p-1} n^{2p-1} + \beta_0. \tag{24}$$

In addition, the assumption above also allows the independent term in (23) to be approximated by $\beta_0 \approx -\frac{\sigma_w^2}{\sigma_v^2}$, since $|\beta'_1| < 1$ as previously stated. Substituting this into (24) and by equating to zero, an approximation for $n_{\mathrm{c}}^{(p)}$ can be provided as

$$n_{\mathrm{c}}^{(p)} \approx \left[\beta_{2p-1}^{-1} \frac{\sigma_w^2}{\sigma_v^2} + \alpha\right]^{\frac{1}{2p-1}} \doteq \tilde{n}_{\mathrm{c}}^{(p)} \tag{25}$$

where we incorporate an additional parameter α to provide a valid solution for the convergence time in case the condition $\sigma_w^2 \gg \sigma_v^2$ is not fulfilled. More precisely, the rationale of α is to lower saturate the approximation in (25) and avoid the meaningless result $\tilde{n}_{\mathrm{c}}^{(p)} = 0$ when $\sigma_w^2/\sigma_v^2 \to 0$. Given that the convergence time fulfills $n_{\mathrm{c}}^{(p)} \geq 1$, the value of α can be computed by forcing the asymptotic of (25) to become

$$\lim_{\frac{\sigma_w^2}{\sigma_v^2} \to 0} \left[\beta_{2p-1}^{-1} \frac{\sigma_w^2}{\sigma_v^2} + \alpha\right]^{\frac{1}{2p-1}} = 1 \tag{26}$$

which leads to $\alpha = 1$.

Even though the complexity of solving $n_{\mathrm{c}}^{(p)}$ for (22) has been considerably reduced, the result in (25) still lacks the knowledge of β_{2p-1} to become a truly closed-form expression for the Kalman filter convergence time. As a matter of fact, β_{2p-1} is the coefficient of the highest order term of $\operatorname{Tr}(\mathbf{M}_n)$ in (20), that is, $\beta_{2p-1} = \beta'_{2p}$, for which an expression can interestingly be obtained in closed form through (19) using matrices $\mathbf{H}_p$, $\mathbf{F}_p$, and $\mathbf{G}_p$ defined in (15)–(17). To this end, we proceed by computing the highest order term of (19) in a step-by-step basis. First, by evaluating the polynomial $\left(\mathbf{H}_p \mathbf{F}_p^{-m} \mathbf{G}_p\right)^2$ for different values of p, it is found that

$$\left[\left(\mathbf{H}_p \mathbf{F}_p^{-m} \mathbf{G}_p\right)^2\right]_{\mathrm{HO}} = \frac{m^{2(p-1)}}{[(p-1)!]^2} \tag{27}$$

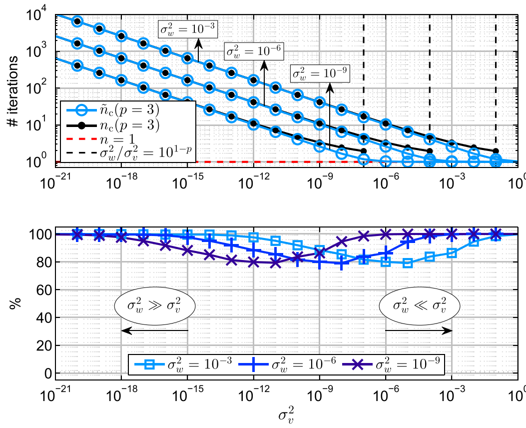

Fig. 1. (Top) Closed-form approximation for the convergence time $\tilde{n}_c^{(p)}$ in (31), versus exact solution $n_c^{(p)}$ for (22), with $p = 3$. (Bottom) Goodness of $\tilde{n}_c^{(p)}$ in terms of the resulting BCRB steady-state performance $\gamma \doteq [\mathbf{J}_\mathrm{B}^{-1}(\infty)]_{1,1} \big/ \big[\mathbf{J}_\mathrm{B}^{-1}(\tilde{n}_c^{(p)})\big]_{1,1}$.

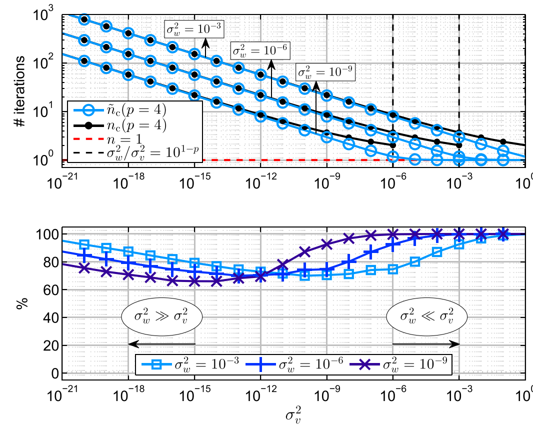

Fig. 2. Same as Fig. 1, but with $p = 4$.

with the subscript HO on the left-hand side denoting the highest order term. Then, the result in (27) is doubly integrated through the inner and outer summatories

$$\left[\sum_{m=1}^{n-k} \frac{m^{2(p-1)}}{[(p-1)!]^2}\right]_{\mathrm{HO}} = \frac{(n-k)^{2p-1}}{(2p-1)\,[(p-1)!]^2} \tag{28}$$

$$\left[\sum_{k=1}^{n} \frac{(n-k)^{2p-1}}{(2p-1)\,[(p-1)!]^2}\right]_{\mathrm{HO}} = \frac{n^{2p}}{2p(2p-1)\,[(p-1)!]^2}. \tag{29}$$

From (29), we can extract the coefficient of interest

$$\beta_{2p-1} = \frac{1}{2p(2p-1)\,[(p-1)!]^2} \tag{30}$$

which, substituted into (25), eventually leads to the following closed-form approximation for the convergence time of a pth-order Kalman filter:

$$\tilde{n}_c^{(p)} = \left[2p(2p-1)\,[(p-1)!]^2\,\frac{\sigma_w^2}{\sigma_v^2} + 1\right]^{\frac{1}{2p-1}}. \tag{31}$$

We recall that a diffuse initialization of the Kalman filter is being considered here. This means that no *a priori* information about the parameters of interest is available to help the Kalman filter converge more rapidly to the steady state. The filter operates with absolute initial uncertainty, thus causing the convergence time to be maximum. Therefore, the result derived in (31) can be understood as an upper bound on the Kalman filter convergence time.

V. Goodness of the Proposed Approximation

This section aims at illustrating the goodness of the approximation for the Kalman filter convergence time proposed in (31) for a broad range of values of σ_w^2 and σ_v^2. Two different plots are analyzed. On the one hand, the top plots of Figs. 1 and 2 show a comparison of the proposed result $\tilde{n}_c^{(p)}$ in (31) to the exact solution $n_c^{(p)}$ given by the real root of (22) for $p = 3$ and $p = 4$, respectively. A very tight match between both lines can be observed for practically all values of σ_w^2 and σ_v^2, even though small discrepancies appear as expected when the assumption $\sigma_w^2 \gg \sigma_v^2$ is not fulfilled. In order to provide a unique real solution for $n_c^{(p)}$, the polynomial in (22) is required to have only one real root. This is found to occur when the process and measurement noise variances meet $\sigma_w^2/\sigma_v^2 \geq 10^{1-p}$. For ratios below this value, the polynomial presents multiple real roots, which hinders the distinction of the correct solution for $n_c^{(p)}$. In this situation, our approximation is lower saturated to $\tilde{n}_c^{(p)} = 1$ as intuitively expected for large σ_v^2, thus confirming that its applicability can safely be extended to all values of σ_w^2 and σ_v^2.

On the other hand, the bottom plots of Figs. 1 and 2 show the goodness of $\tilde{n}_c^{(p)}$ in terms of the resulting BCRB steady-state performance, measured through the ratio $\gamma \doteq [\mathbf{J}_\mathrm{B}^{-1}(\infty)]_{1,1} \big/ \big[\mathbf{J}_\mathrm{B}^{-1}(\tilde{n}_c^{(p)})\big]_{1,1}$. It is observed that the smaller σ_v^2, the tighter the matching of $\big[\mathbf{J}_\mathrm{B}^{-1}(\tilde{n}_c^{(p)})\big]_{1,1}$ to $[\mathbf{J}_\mathrm{B}^{-1}(\infty)]_{1,1}$, and hence, the better the approximation in (31), leading to $\gamma \to 100\%$. As σ_v^2 gradually increases and the assumption $\sigma_w^2 \gg \sigma_v^2$ gets compromised, γ worsens, but just slightly below 80% and 70%, as shown in Figs. 1 and 2, respectively. This manifests the limited operation range of the proposed approach, even though an acceptable result can still be provided. When σ_v^2 is too large and $\sigma_w^2 \gg \sigma_v^2$ no longer holds, the lower saturation in (26) comes into action, and interestingly, $\gamma \to 100\%$ again. Note that a too large process noise, though, indicates that the Kalman filter state-space model does not quite well fit the incoming measurements, thus causing the former to become ill-posed and consequently not to be able to perform filtering. The result of γ above confirms that in this situation, there is no point in iterating the BCRB further than the very first iteration, as it provides reasonably no benefit in terms of performance enhancement.

VI. Conclusion

In this letter, a novel closed-form approximated upper bound for the convergence time of Kalman filters of any order has been presented. The expression has been obtained by reformulating the Kalman filtering problem in batch mode and analyzing the corresponding FIM. The goodness of the approach has been confirmed through simulation results. The proposed contribution provides a direct relationship of the Kalman filter convergence time with the parameters playing a key role in its behavior. Hence, the provided result becomes a valuable tool that removes the need to resort to numerical evaluations or Monte Carlo simulations for tuning Kalman filter-based applications in practice.

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