Asymptotic Analysis of the Convergence Time of Autoregressive Kalman Filters

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***Abstract*—In recent years, the Kalman filter has become the prime approach for estimating parameters that evolve following some dynamic model and prior statistics. In addition, recent contributions are introducing the use of autoregressive models in the state-space formulation to deal with correlated Gaussian-distributed magnitudes. However, the derivation of closed-form expressions for predicting their performance during the design stage is still an open problem. In that regard, in this letter we derive novel approximate closed-form upper bounds to characterize the convergence time of autoregressive Kalman filters. To this end, we extend a batch mode-based approach previously proposed in the literature that reveals the need for a dedicated dual-asymptotic analysis for this kind of techniques. Simulations are provided to show the goodness of the derived results.**

***Index Terms*—Autoregressive process, Bayesian Cramér-Rao bound, convergence time, Kalman filter, steady state.**

I. Introduction

THE use of Kalman filter-based techniques for providing the minimum mean square error estimate of dynamically-evolving system parameters is becoming increasingly widespread in practice. Recent contributions are showing a great interest in designing filters to track magnitudes that can be modeled using some correlated Gaussian function, such as the class of autoregressive (AR) random processes. This is the case of [1]–[3], where a Kalman filter with an AR model is employed for channel estimation and equalization, as well as for maneuvering target tracking [4] and signal dereverberation [5]. Some more examples can be found in carrier tracking in global navigation satellite system receivers [6], [7]. In there, ionospheric scintillation disturbances are estimated using an AR state-space formulation that is thereafter hybridized with a kinematic one [8] to eventually provide clean carrier phase and frequency estimates. This hybridization concept can also be applied to the problem of joint frequency and phase noise estimation in crystal oscillators [9], where a kinematic model is considered for the former while the latter can be synthesized using a first-order AR process, namely AR(1) [10].

From the design point of view, one feature of great practical interest is the ability to determine the expected performance of the above techniques in a straightforward manner. That is, by using closed-form expressions that depend on the tuning parameters only, and thus avoiding the need for cumbersome evaluations or empirical assessments of the filter through Monte Carlo iterations. The information about the estimation performance is provided by the so-called Bayesian Cramér-Rao bound (BCRB) given by the inverse of the Bayesian information matrix (BIM) [11]. However, it is well known that it has the form of a discrete-time Riccati equation for which a closed-form solution is difficult to obtain.

A number of efforts to address this problem can be found in the literature. For instance, [12] and [13] derive closed-form expressions for kinematic and AR models, respectively, but they are restricted to the availability of an accurate prior or the presence of small dynamics and model uncertainty [14], often unrealistic in practice. Moreover, these works focus on the Kalman filter steady state experienced when process noise is present [15], whereas the emphasis placed on the precedent convergence period is, to our best, scarce. An expression for its performance is derived in [16], even though no clue about its duration is given. This latter point is addressed by the authors of this letter in [17], [18] for kinematic models, whereas the analysis for AR ones remains, to date, unsolved.

This letter intends to shed some light on the BCRB behavior for this kind of techniques, with the focus on determining the convergence time toward the steady state. The proposal is to apply the approach presented in [18], wherein we derive the corresponding batch-mode Fisher information matrix (FIM) that results from evaluating the Kalman filtering problem from the best linear unbiased estimator (BLUE) standpoint. The beauty of the approach is that, interestingly, the BIM coincides with the FIM when considering a diffuse initialization of the filter (i.e. noninformative prior) [19], meaning that the FIM, for which an expression to work with is available, can thence be used to unveil the insights of the Kalman filter performance. As a matter of fact, this approach has been proven effective for deriving approximate closed-form upper bounds for the convergence time of kinematic filters. Nevertheless, the application to AR ones motivated by their practical interest as highlighted in [1]–[7] reveals the need for an extended analysis devoted to this very specific problem.

Therefore, this letter contributes with the derivation of closed-form upper bounds for the convergence time of *autoregressive* Kalman filters, denoted in the sequel as KF-AR. To this end, we consider the generic approach of a hybrid filter that combines a given state-space model with that of an AR process, from which a non-hybrid filter encompassing only the latter can subsequently be particularized. In Section III we summarize the batch-mode formulation that corresponds to this technique. Then, a dual characterization of the convergence time through asymptotic analysis is performed to obtain the results for each filter. To

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align our work with the contributions in the literature, we focus on second-order kinematic models, as they are representative of plenty of applications in practice, combined with AR(1) models. This leads to a three-state hybrid KF-AR hereinafter denoted as KF(2)-AR(1), and a one-state non-hybrid filter denoted as KF-AR(1). Note that the analysis could ultimately be extended to higher model orders, even though the derivations are mathematically not quite tractable, and thus remain out of the scope of this letter.

II. Formulation of KF-AR Equations

A. Kalman Filter State-Space and Observation Models

Consider first a linear second-order Kalman filter with the following discrete-time state transition equation,

$$\mathbf{x}_v(n+1) = \mathbf{F}_v \mathbf{x}_v(n) + \mathbf{G}_v v(n) \tag{1}$$

where $\mathbf{x}_v(n)$ is the two-dimensional state vector containing the sample functions of the parameters of interest at time sample n, and $\mathbf{F}_v$ is the (2×2) transition matrix used to propagate $\mathbf{x}_v(n)$ toward $\mathbf{x}_v(n+1)$. The term $v(n)$ is the process noise accounting for the possible mismatch of the Kalman state-space model to the incoming data. It is usually modeled as $v(n) \sim \mathcal{N}(0, \sigma_v^2)$, with $\mathrm{E}[v(i)v^*(j)] = 0$ for $i \neq j$, and degrades the Kalman states through $\mathbf{G}_v$.

On the other hand, the Kalman filter provides an estimate of $\mathbf{x}_v(n)$ based on the incoming scalar noisy measurements,

$$z_v(n) = \mathbf{H}_v \mathbf{x}_v(n) + w(n) \tag{2}$$

with $\mathbf{H}_v$ the (1×2) observation matrix, and $w(n)$ is the measurement noise corrupting the observations, usually modeled as $w(n) \sim \mathcal{N}(0,\sigma_w^2)$, with $\mathrm{E}[w(i)w^*(j)] = 0$ for $i \neq j$. The process and measurement noises are independent zero-mean Gaussian disturbances, that is, $\mathrm{E}[v(i)w^*(j)] = \mathrm{E}[w(i)v^*(j)] = 0$ for any i and j.

B. Signal Model for AR(1) Process

Let $\psi(n)$ be the sample function of an AR(1) process whose discrete-time evolution is expressed as,

$$\psi(n+1) = \beta\psi(n) + s(n) \tag{3}$$

with β the AR coefficient, and $s(n) \sim \mathcal{N}(0,\sigma_s^2)$ is the so-called AR driving noise [19], with $\mathrm{E}[s(i)s^*(j)] = 0$ for $i \neq j$. Interestingly, the model in (3) presents the same structure as the state transition equation in (1), where $\psi(n)$ propagates toward $\psi(n+1)$ through the coefficient β, and $s(n)$ plays the role of the Kalman process noise. This information is used to formulate the hybrid KF-AR next.

C. The Hybrid Autoregressive Kalman Filter: KF(2)-AR(1)

The AR(1) state-space model in (3) can be merged into the second-order Kalman filter architecture introduced in Section II-A. This gives rise to the so-called KF(2)-AR(1), a three-state Kalman filter with the following state transition equation,

$$\begin{bmatrix}\mathbf{x}_v(n+1)\\ \psi(n+1)\end{bmatrix} = \mathbf{F}\begin{bmatrix}\mathbf{x}_v(n)\\ \psi(n)\end{bmatrix} + \mathbf{G}\mathbf{q}(n) \tag{4}$$

where we define $\mathbf{x}(n) \doteq \begin{bmatrix}\mathbf{x}_v(n) & \psi(n)\end{bmatrix}^T$ as the augmented state vector, and $\mathbf{q}(n) \doteq \begin{bmatrix}v(n) & s(n)\end{bmatrix}^T$ with $\mathrm{E}[v(i)s^*(j)] = \mathrm{E}[s(i)v^*(j)] = 0$ for any i and j. The augmented transition $\mathbf{F}$ and process noise propagation $\mathbf{G}$ matrices become $\mathbf{F} \doteq \begin{bmatrix}\mathbf{F}_v & \mathbf{0}_{2\times1}\\ \mathbf{0}_{1\times 2} & \beta\end{bmatrix}$ and $\mathbf{G} \doteq \begin{bmatrix}\mathbf{G}_v & \mathbf{0}_{2\times1}\\ \mathbf{0}_{1\times 2} & 1\end{bmatrix}$.

On the other hand, the KF(2)-AR(1) estimates $\mathbf{x}(n)$ from the following scalar measurements,

$$z(n) = z_v(n) + \psi(n) = \mathbf{H}\mathbf{x}(n) + w(n) \tag{5}$$

from which the augmented measurement matrix becomes $\mathbf{H} \doteq \begin{bmatrix}\mathbf{H}_v & 1\end{bmatrix}$. The noises $w(n)$ and $s(n)$ are independent zero-mean Gaussian disturbances, meaning that $\mathrm{E}[s(i)w^*(j)] = \mathrm{E}[w(i)s^*(j)] = 0$ for any i and j.

III. Batch Formulation of the KF(2)-AR(1)

A. Batch-Mode Estimates and Performance Lower Bound

By stacking all the measurements $z(n)$ in the data record into a vector $\mathbf{z}_n$, the KF(2)-AR(1) can be formulated in batch mode as [17],

$$\mathbf{z}_n = \mathbf{A}_n \mathbf{x}(n) + \mathbf{B}_n \mathbf{u}_n \tag{6}$$

where $\mathbf{A}_n$ and $\mathbf{B}_n$ are $(n\times 3)$ and $(n \times (3n-2))$ matrices given by,

$$\mathbf{A}_n \doteq \begin{bmatrix}\mathbf{H}\mathbf{F}^{-(n-1)}; & \mathbf{H}\mathbf{F}^{-(n-2)}; & \cdots; & \mathbf{H}\end{bmatrix} \tag{7}$$

$$\mathbf{B}_n \doteq \begin{bmatrix}\mathbf{I}_n, & \mathbf{B}_{n,v}, & \mathbf{B}_{n,s}\end{bmatrix} \tag{8}$$

with $\mathbf{I}_n$ the n-dimensional identity matrix, and

$$\mathbf{B}_{n,v} \doteq \begin{bmatrix}-\mathbf{H}_v\mathbf{F}_v^{-1}\mathbf{G}_v & \cdots & -\mathbf{H}_v\mathbf{F}_v^{-(n-1)}\mathbf{G}_v\\ 0 & \cdots & -\mathbf{H}_v\mathbf{F}_v^{-(n-2)}\mathbf{G}_v\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 0\end{bmatrix} \tag{9}$$

$$\mathbf{B}_{n,s} \doteq \begin{bmatrix}-\beta^{-1} & \cdots & -\beta^{-(n-1)}\\ 0 & \cdots & -\beta^{-(n-2)}\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 0\end{bmatrix}. \tag{10}$$

In (6), $\mathbf{u}_n \doteq \begin{bmatrix}\mathbf{w}_n, & \mathbf{v}_n, & \mathbf{s}_n\end{bmatrix}^T$ is a $((3n-2)\times 1)$ vector containing the measurement and process noise samples, with

$$\mathbf{w}_n \doteq \begin{bmatrix}w(1) & w(2) & \cdots & w(n)\end{bmatrix} \tag{11}$$

$$\mathbf{v}_n \doteq \begin{bmatrix}v(1) & v(2) & \cdots & v(n-1)\end{bmatrix} \tag{12}$$

$$\mathbf{s}_n \doteq \begin{bmatrix}s(1) & s(2) & \cdots & s(n-1)\end{bmatrix}. \tag{13}$$

When considering a diffuse initialization of the filter and using the BLUE to determine the optimal minimum variance unbiased estimate of $\mathbf{x}(n)$, the estimation performance is lower bounded by the inverse of the following FIM,

$$\mathbf{J}(n) = \mathbf{A}_n^H \boldsymbol{\Sigma}_{\mathbf{z}_n} \mathbf{A}_n \tag{14}$$

with $\boldsymbol{\Sigma}_{\mathbf{z}_n}$ the covariance matrix of the linear Gaussian measurements $\mathbf{z}_n \sim \mathcal{N}(\mathbf{A}_n\mathbf{x}(n), \boldsymbol{\Sigma}_{\mathbf{z}_n})$ in (6),

$$\boldsymbol{\Sigma}_{\mathbf{z}_n} \doteq \mathrm{E}\left[\mathbf{B}_n\mathbf{u}_n\mathbf{u}_n^H\mathbf{B}_n^H\right] = \mathbf{B}_n\boldsymbol{\Sigma}_{\mathbf{u}_n}\mathbf{B}_n^H \tag{15}$$

where $\boldsymbol{\Sigma}_{\mathbf{u}_n} \doteq \mathrm{diag}(\begin{bmatrix}\boldsymbol{\sigma}^2_{w,(1\times n)} & \boldsymbol{\sigma}^2_{v,(1\times(n-1))} & \boldsymbol{\sigma}^2_{s,(1\times(n-1))}\end{bmatrix})$.

After obtaining the expression of the FIM in (14), the missing step is the evaluation of the measurement covariance matrix $\boldsymbol{\Sigma}_{\mathbf{z}_n}$ in (15), which contains, particularly in its diagonal elements, the information of the convergence time.

B. Inner Structure of the Measurement Covariance Matrix

By exploiting the knowledge of $\mathbf{B}_n$ and $\boldsymbol{\Sigma}_{\mathbf{u}_n}$, matrix $\boldsymbol{\Sigma}_{\mathbf{z}_n}$ for the problem at hand is found to be formed by three terms,

$$\boldsymbol{\Sigma}_{\mathbf{z}_n} = \sigma_v^2\mathbf{M}_n + \sigma_s^2\mathbf{S}_n + \sigma_w^2\mathbf{I}_n \tag{16}$$

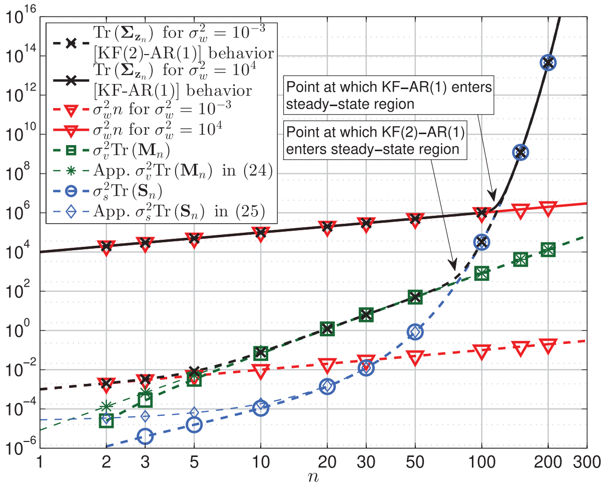

Fig. 1. Asymptotic evolution of $\mathrm{Tr}(\Sigma_{\mathbf{z}_n})$ versus the constituent elements of $\Sigma_{\mathbf{z}_n}$ in (16), for $\sigma_w^2 = \{10^{-3}, 10^{4}\}$ to simulate the KF(2)-AR(1) and KF-AR(1), respectively.

which resembles the expression obtained in [18, eq. (10)] with the addition of a new term $\sigma_s^2 \mathbf{S}_n$ owing to the AR contribution. $\mathbf{M}_n$ and $\mathbf{S}_n$ are nonnegative matrices whose diagonal elements are,

$$[\mathbf{M}_n]_{k,k} = \sum_{m=1}^{n-k} \left(\mathbf{H}_v \mathbf{F}_v^{-m} \mathbf{G}_v\right)^2 \tag{17}$$

$$[\mathbf{S}_n]_{k,k} = \sum_{m=1}^{n-k} \beta^{-2m} \tag{18}$$

which are to be used for the derivations later in Section IV.

C. Second-Order Kinematic State-Space Model

The problem has been so far formulated for Kalman filters hybridizing an AR(1) process with any generic two-dimensional state-space model. However, in this letter we consider the latter to be a time-varying magnitude with discrete-time evolution driven by a time-invariant second-order kinematic process that adopts the following state-space model,

$$\begin{bmatrix}\theta(n)\\ \dot{\theta}(n)\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\theta(n-1)\\ \dot{\theta}(n-1)\end{bmatrix} + \begin{bmatrix}1/2\\ 1\end{bmatrix} v(n) \tag{19}$$

from which we define $\mathbf{x}_v(n) \doteq \begin{bmatrix}\theta(n) & \dot{\theta}(n)\end{bmatrix}^T$, $\mathbf{F}_v \doteq \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ and $\mathbf{G}_v \doteq \begin{bmatrix}1/2 & 1\end{bmatrix}^T$. The observations are given by,

$$z(n) = \theta(n) + w(n) \tag{20}$$

and thus by direct comparison with (2), $\mathbf{H}_v \doteq \begin{bmatrix}1 & 0\end{bmatrix}$.

IV. Asymptotic Analysis of the KF-AR Convergence Time

The information of the KF-AR convergence time can be extracted by observing the evolution of the trace of $\Sigma_{\mathbf{z}_n}$ over n. The latter is depicted in Fig. 1, which also includes the evolution of the three separate components in (16). For illustrative purposes, the values $\beta = 0.9$, $\sigma_s^2 = 10^{-6}$ and $\sigma_v^2 = 10^{-4}$ are used. From Fig. 1 the results for the KF(2)-AR(1) and KF-AR(1) can be derived through a measurement noise dual characterization-based analysis. On the one hand, the behavior of the KF(2)-AR(1) is evaluated by considering small measurement noise, $\sigma_w^2 = 10^{-3}$. On the other hand, the behavior of the KF-AR(1) can be readily simulated by considering the KF(2)-AR(1) with sufficiently large measurement noise so as to neglect the kinematic module, $\sigma_w^2 = 10^4$.

A. Asymptotic Closed-Form Approximation for KF(2)-AR(1)

For small n, the trace of $\Sigma_{\mathbf{z}_n}$ is dominated by the contribution of σ_w^2. At some point, a change of trend is produced and the component of σ_v^2 starts being dominant. This point determines the Kalman convergence time when in the absence of the AR module [18]. However, in the KF(2)-AR(1) a new change of state occurs later, in which the AR contribution given by σ_s^2 becomes the dominating one, a situation that remains then for $n \to \infty$. Hence, this becomes the turning point at which the KF(2)-AR(1) enters the steady-state region. As observed in Fig. 1, this occurs when the contributions of σ_v^2 and σ_s^2 intersect. Therefore, the convergence time for the KF(2)-AR(1), denoted henceforth as $n_{\mathrm{c}}^{(\mathrm{h})}$ with the superscript (h) standing for *hybrid*, can be found as the real solution for n to the following equality,

$$\sigma_v^2 \mathrm{Tr}\left(\mathbf{M}_n\right) = \sigma_s^2 \mathrm{Tr}\left(\mathbf{S}_n\right). \tag{21}$$

The problem of solving (21) requires determining the traces of $\mathbf{M}_n$ and $\mathbf{S}_n$. To this end, we fortunately have (17) and (18) to work with. Using the matrices $\mathbf{H}_v$, $\mathbf{F}_v$ and $\mathbf{G}_v$ introduced in Section III-C, the former can be found as in [17, eq. (23)],

$$\mathrm{Tr}\left(\mathbf{M}_n\right) = \frac{1}{24}\left(2n^4 - 4n^3 + n^2 + n\right) \tag{22}$$

whereas the latter can be computed by summing the elements in (18) for $k \doteq [1..n]$. This results in a set of geometric series that, after some mathematical manipulations, equals to,

$$\mathrm{Tr}\left(\mathbf{S}_n\right) = n\left(\frac{1}{1-\beta^{-2}} - 1\right) - \frac{\beta^{-2}}{\left(1-\beta^{-2}\right)^2}\left(1-\beta^{-2n}\right). \tag{23}$$

As can be observed, expressions (22) and (23) as such pose a considerable mathematical difficulty for deriving a closed-form solution for $n_{\mathrm{c}}^{(\mathrm{h})}$. For this reason, we propose an approximation of reduced complexity that relies on the following observations. As a general rule, the convergence time of a Kalman filter meets $1 \leq n_{\mathrm{c}} < \infty$. Being this raised to the fourth power in (22), the trace of $\mathbf{M}_n$ becomes rapidly dominated by the highest-order term of the polynomial,

$$\mathrm{Tr}\left(\mathbf{M}_n\right) \approx \frac{n^4}{12}. \tag{24}$$

In addition, the AR coefficient β must fulfill $|\beta| < 1$ for an AR(1) process to be stable. As a result of both observations, the trace of $\mathbf{S}_n$ in (23) is empirically found to be dominated by the second term on the right-hand side, where one can also resort to $(1 - \beta^{-2n}) \approx -\beta^{-2n}$, thus boiling down to,

$$\mathrm{Tr}\left(\mathbf{S}_n\right) \approx \frac{\beta^{-2(n+1)}}{\left(1-\beta^{-2}\right)^2}. \tag{25}$$

By substituting (24) and (25) into (21) and solving for n, an approximation for $n_{\mathrm{c}}^{(\mathrm{h})}$ can be provided as,

$$n_{\mathrm{c}}^{(\mathrm{h})} \approx \frac{2W_{-1}\left(-\left(\frac{3\ln(\beta)^4 \beta^2 \sigma_s^2}{4\sigma_v^2(\beta^4 - 2\beta^2 + 1)}\right)^{\frac{1}{4}}\right)}{\ln\left(\beta\right)} \tag{26}$$

where $W_{-1}(\cdot)$ refers to the -1 real branch of the so-called Lambert *W* function. It is defined in the interval $[-\exp(-1), 0)$ [20]. Hence, the result in (26) provides a valid solution as long as the following relationship is fulfilled,

$$0 < \sigma_s^2 \leq \frac{4\sigma_v^2\left(\beta^4 - 2\beta^2 + 1\right)}{3\ln(\beta)^4\beta^2\exp(4)}. \quad (27)$$

After some polynomial fitting analysis we find that $W_{-1}(x)$ in the region of interest can be approximated by,

$$W_{-1}(x) \approx 2.4\log_{10}(-x) - 2.24. \quad (28)$$

Therefore, by substituting x with the argument of $W_{-1}(\cdot)$ in (26), an approximation for the KF(2)-AR(1) convergence time can be provided in form of closed-form upper bound as,

$$n_{\mathrm{c}}^{(\mathrm{h})} \approx \frac{4.8\log_{10}\left(\left(\frac{3\ln(\beta)^4\beta^2\sigma_s^2}{4\sigma_v^2(\beta^4-2\beta^2+1)}\right)^{\frac{1}{4}}\right) - 4.48}{\ln(\beta)} \doteq \tilde{n}_{\mathrm{c}}^{(\mathrm{h})}. \quad (29)$$

B. Asymptotic Closed-Form Approximation for KF-AR(1)

For the non-hybrid filter, $\mathrm{Tr}(\mathbf{\Sigma}_{\mathbf{z}_n})$ is also dominated by the contribution of σ_w^2 for small n. Then, a change of state is produced when intersecting with the contribution of σ_s^2, which dominates again in the steady state (i.e. $n \rightarrow \infty$). This can be observed in Fig. 1, where the effect of σ_v^2 becomes neglected here, in contrast to the hybrid filter. Therefore, the convergence time for the KF-AR(1), denoted henceforth as $n_{\mathrm{c}}^{(\mathrm{nh})}$ with the superscript (nh) standing for *non-hybrid*, can be found as the real solution for n to the equality,

$$\sigma_s^2\mathrm{Tr}(\mathbf{S}_n) = \sigma_w^2 n. \quad (30)$$

At this point we can take advantage of the approximation for $\mathrm{Tr}(\mathbf{S}_n)$ in (25). Substituted into (30) and solving for n, an approximation for $n_{\mathrm{c}}^{(\mathrm{nh})}$ can be provided as,

$$n_{\mathrm{c}}^{(\mathrm{nh})} \approx \frac{W_{-1}\left(\frac{2\ln(\beta)\beta^2\sigma_s^2}{\sigma_w^2(\beta^4-2\beta^2+1)}\right)}{2\ln(\beta)} \quad (31)$$

which thus provides a valid solution when,

$$0 < \sigma_s^2 \leq \frac{\sigma_w^2\left(2\beta^2 - \beta^4 - 1\right)}{2\ln(\beta)\beta^2\exp(1)}. \quad (32)$$

By making use of the $W_{-1}(x)$ function approximation in (28) and substituting x in (28) with the argument of $W_{-1}(\cdot)$ in (31), an approximation for the KF-AR(1) convergence time can be provided in form of closed-form upper bound as,

$$n_{\mathrm{c}}^{(\mathrm{nh})} \approx \frac{1.2\log_{10}\left(\frac{2\ln(\beta)\beta^2\sigma_s^2}{\sigma_w^2(2\beta^2-\beta^4-1)}\right) - 1.12}{\ln(\beta)} \doteq \tilde{n}_{\mathrm{c}}^{(\mathrm{nh})}. \quad (33)$$

V. Goodness of the Proposed Approximations

This section aims at illustrating the goodness of the approximations for the KF-AR convergence time upper bounds in (29) and (33). For this purpose, Fig. 2 depicts the matching of our results with the empirical convergence times, measured through the metrics $\Delta\tilde{n}_{\mathrm{c}}^{(\mathrm{h})} \doteq (\tilde{n}_{\mathrm{c}}^{(\mathrm{h})} - n_{\mathrm{c}}^{(\mathrm{h})})/n_{\mathrm{c}}^{(\mathrm{h})}$ and $\Delta\tilde{n}_{\mathrm{c}}^{(\mathrm{nh})} \doteq (\tilde{n}_{\mathrm{c}}^{(\mathrm{nh})} - n_{\mathrm{c}}^{(\mathrm{nh})})/n_{\mathrm{c}}^{(\mathrm{nh})}$ as the error of the approximations, normalized to the empirical values. We consider different values of σ_w^2 and σ_s^2, while the values of β and σ_v^2 remain as in Fig. 1. The terms $n_{\mathrm{c}}^{(\mathrm{h})}$ and $n_{\mathrm{c}}^{(\mathrm{nh})}$ are computed as the time sample for which the BCRB reaches $[\mathbf{J}^{-1}(\infty)]_{i,i}/[\mathbf{J}^{-1}(n_{\mathrm{c}}^{(\cdot)})]_{i,i} = 0.99$, with $i = 3$ for the KF(2)-AR(1) and $i = 1$ for the KF-AR(1), and 0.99 a conservative value. In the figure, positive values indicate an overestimated convergence time, meaning that the BCRB reached the steady state some iterations ago. On the contrary, negative values denote an underestimation where the BCRB is still to reach the steady state. Since we are dealing with upper-bound approximations, overestimation is preferred to underestimation. As can be observed, a tight match between our results and the empirical convergence times can be observed, with the previous metrics very close to the ideal zero in the direction of overestimation for the hybrid filter and very little underestimation for the non-hybrid one. The approximations tend to the empirical values when increasing σ_w^2, that is, when the relationship between the latter and the process noise variances is such that it makes the presence of a convergence period toward some steady state evident. Following this reasoning, discrepancies appear when σ_s^2 gets much smaller than σ_w^2, thus causing the steady-state region to disappear. As a consequence, it becomes more difficult to properly characterize a convergence time that, conceptually, loses any sense of being in this situation, thus escaping from the application coverage that our results are intended for.

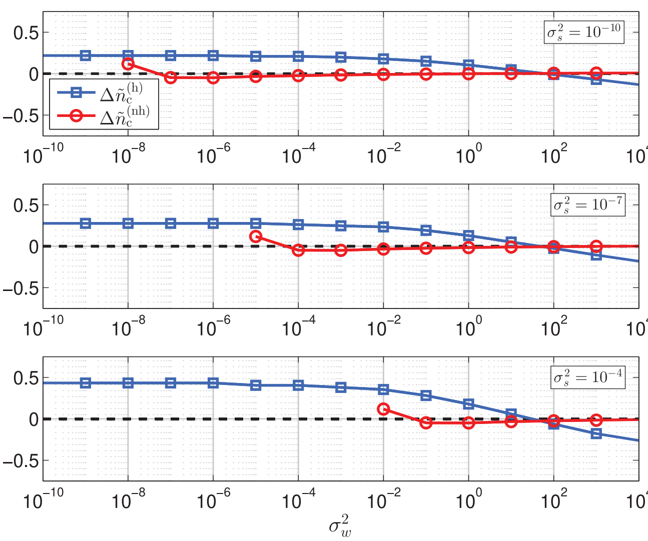

Fig. 2. Errors of asymptotic approximations normalized to empirical convergence time, $\Delta\tilde{n}_{\mathrm{c}}^{(\mathrm{h})}$ and $\Delta\tilde{n}_{\mathrm{c}}^{(\mathrm{nh})}$.

VI. Conclusion

In this letter, novel approximate closed-form upper bounds for the convergence time of autoregressive Kalman filters have been derived. By means of a dual-asymptotic analysis of the batch-mode Fisher information matrix, expressions for both hybrid and non-hybrid filters have been obtained. As expected, the results have been shown effective for their intended purpose, when the process noise induces the presence of a steady-state region that raises sense to the concept of convergence period. The proposed contribution allows predicting the convergence time of such filters as a function of the parameters of the problem playing a key role in their behavior. Therefore, the provided results become a valuable tool for the practical tuning and design of autoregressive Kalman filter-based applications in a straightforward manner.

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