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Optimal Fractional Non-Coherent Detector for High-Sensitivity GNSS Receivers Robust against Residual Frequency Offset and Unknown Bits

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***Abstract*—This paper addresses the problem of finding the optimal non-coherent detector or Post Detection Integration (PDI) technique to acquire weak signals in the context of High-Sensitivity Global Navigation Satellite System (HS-GNSS) receivers. This detector is derived using the Generalized Likelihood Ratio Test (GLRT) in the presence of data bits and variations in the carrier phase. The resulting detector is difficult to implement in practice because the amplitude of the signal must be known a priori. Two approximations of the resulting detector, which depend on the signal-to-noise ratio, are proposed, namely the Non-coherent PDI (NPDI) and non-quadratic NPDI ($\mathbf{NPDI_{nq}}$) techniques. From this result, we prove that in general the $\mathbf{NPDI_{nq}}$ technique is the best option to detect weak signals in HS-GNSS receivers. In addition, a new statistical characterization of the $\mathbf{NPDI_{nq}}$ technique is proposed, which improves the approach used in the literature by applying the central limit theorem.**

***Index Terms*—Detection threshold, fractional exponent, GLRT, HS-GNSS receivers, PDI techniques.**

I. Introduction

Weak signal conditions are certainly the main obstacle in High-Sensitivity Global Navigation Satellite system (HS-GNSS) receivers since the severe attenuation makes it difficult to detect the received signal [1]. In this situation, the increase of the coherent integration time is the optimal way to reduce the effect of noise so that the signal can be detected more easily. Nonetheless, the coherent integration time duration is limited in practice by the followings uncertainties: residual frequency offset, data bits, and phase noise.

The way to circumvent this limitation is to adopt Post Detection Integration (PDI) techniques or non-coherent accumulations, which are more robust against frequency offset, data bits, and phase noise than the coherent integration. Although PDI techniques suffer some degradation in the signal detection performance with respect to the ideal coherent integration (without taking into account the impairments that limit the coherent integration duration), the use of PDI techniques is the only choice we have if long integration times need to be implemented [2]. Thereby, PDI techniques have become an indispensable tool to acquire weak signals in the context of HS-GNSS receivers.

HS-GNSS receiver architectures use advanced signal processing techniques to improve the acquisition of weak signals, resorting to increased correlation intervals through the combination of coherent integrations and PDI techniques [3]. Weak GNSS signals are detected by exploiting long coherent integration times and adopting an optimal number of non-coherent accumulation [4].

The PDI technique most commonly used in GNSS is the Non-coherent PDI (NPDI), which is robust against the limitation arising from the coherent integration such as frequency offset and data bits [5]. Another alternative is the Differential PDI (DPDI) technique, which might lead to a better detection probability than the NPDI technique in absence of data bits. Nevertheless, it suffers a significant degradation in presence of data bits. Alternatively, a key technique that is often dismissed is the non-quadratic NPDI ($\mathrm{NPDI_{nq}}$), which is robust against residual frequency offset, phase noise, and data bits. The drawback of the $\mathrm{NPDI_{nq}}$ technique is that its statistical distribution is not known in a closed-form and it makes more difficult to set a threshold to determinate if the signal is present or not.

However, although several PDI techniques have been proposed, the question about what is the optimal PDI technique to acquire weak signals under practical conditions in HS-GNSS remains open. The purpose of this paper is twofold. Firstly, finding the best PDI technique in presence of several impairments such as the data bits and the variations produced in the carrier phase of the received signal due to the frequency offset. To do so, we derive the Generalized Likelihood Ratio Test (GLRT), which leads to a trade off between the NPDI and $\mathrm{NPDI_{nq}}$ techniques. Secondly, we propose a new statistical characterization of the $\mathrm{NPDI_{nq}}$ metric, which improves the approximation of the Central Limit Theorem (CLT) usually employed in the literature.

II. Signal Model

The main task of the acquisition stage in HS-GNSS receivers is to perform a correlation between the received signal from different satellites with a local replica of the signal transmitted by one satellite. This correlation is computed using several trial values of time-delay and Doppler frequency in order to know if the satellite is in view for given values of time-delay and Doppler frequency or the satellite is absent. The resulting correlation is the cross ambiguity function (CAF) or output of the coherent correlation. This process is usually done for all the satellites, but in this paper, we focus on acquiring only a satellite because it is enough to analyse the detection problem.

HS-GNSS receivers use long correlation intervals to detect weak signals in two steps. First, the CAF is calculated by performing the correlation between the local replica and the received signal using an integration time as long as possible. Second, several CAFs are combined non-coherently by the application of a PDI technique to avoid the cancellation of the signal due to the residual frequency offset and data bits. The result of this process is the output detection metric, which can be expressed as

$$Z = \sum_{k=1}^{N_{nc}} h(y_k), \tag{1}$$

where $k = 1, \ldots, N_{nc}$ and N_{nc} is the number of non-coherent combinations, y_k is the CAF in the instant k evaluated for a value of time-delay and Doppler frequency, and $Z = h(y_k)$ is a non-linear transformation of y_k. It should be added that there is a value of Z for each trial value of time-delay and Doppler frequency, but we omit this dependence since we can consider we are performing the analysis only for one of these values. The problem of detecting signals can be modelled under two hypotheses H_0 and H_1 since the satellite can be in view or not.

- Under H_0: $y_k = n_k$ is a complex Gaussian noise with mean zero and variance σ^2 (the signal from the satellite is absent).
- Under H_1: $y_k = Ad_k e^{j\phi_k} + n_k$ is the signal plus complex Gaussian noise (the signal from the satellite is present).

where d_k is a uniform random variable that contains the unknown data bits taking values -1 or 1 and A is a constant amplitude affected by an unknown phase ϕ_k. The discrimination between the hypotheses H_0 and H_1 is determined by comparing the maximum value of several Z obtained from different trial values of time-delay and Doppler frequency with a given detection threshold. If the maximum value of Z surpasses the value of the threshold, the satellite is assumed to be present. However, if the maximum magnitude of Z does not exceed the detection threshold, the satellite is considered to be absent.

III. Generalized likelihood ratio test

In this subsection, we derive the GLRT assuming that the incoming signal contains data bits of the GNSS signal and that the phase of the signal can change for different time instants. To do so, we have to calculate the likelihood ratio test and replacing the unknown phase by its Maximum Likelihood (ML) estimate [6]. A related approach was used in [7], but they used a ML estimator assuming that the phase of the received signal does not change during the whole observation interval. This causes that the detector in [7] is not robust against the frequency offset. However, in this paper, we want to find the optimal detector being robust against the frequency offset and data bits.

The probability density function (pdf) of the CAF y_k, assuming that it is affected by data bits uniformly distributed taking values 1 and -1, is expressed under H_1 as

$$\begin{aligned} &p(\mathbf{y}; H_1, \phi_k) = \\ &\frac{1}{(\pi\sigma^2)^{N_{nc}}} \exp\left(-\sum_{k=1}^{N_{nc}} \frac{1}{\sigma^2}(I_k^2 + Q_k^2 + A^2)\right) \\ &\prod_{k=1}^{N_{nc}} \cosh\left(\frac{2A}{\sigma^2}(I_k \cos(\phi_k) + Q_k \sin(\phi_k))\right), \end{aligned} \tag{2}$$

where $I_k = \Re(y_k)$, $Q_k = \Im(y_k)$ and $\mathbf{y} = [y_1, \ldots, y_{N_{nc}}]^T$. Under H_0, the pdf can be written as

$$p(\mathbf{y}; H_0) = \frac{1}{(\pi\sigma^2)^{N_{nc}}} \exp\left(-\sum_{k=1}^{N_{nc}} \frac{1}{\sigma^2}(I_k^2 + Q_k^2)\right). \tag{3}$$

The detector is derived by invoking the GLRT, which consist in applying the likelihood ratio test and replacing the unknown parameter with its ML estimate as follows,

$$\Lambda(\mathbf{y}) = \frac{p(\mathbf{y}; H_1, \hat{\phi}_k)}{p(\mathbf{y}; H_0)} \lessgtr \tilde{\gamma}, \tag{4}$$

where $\tilde{\gamma}$ is the detection threshold, $\Lambda(\mathbf{y})$ is the likelihood ratio test and $\hat{\phi}_k$ is the ML estimate of ϕ_k. Substituting (3) and (2) into (4) and including the constants terms in a new threshold γ', the likelihood ratio test can be rewritten as

$$\Lambda'(\mathbf{y}) = \prod_{k=1}^{N_{nc}} \cosh\left(\frac{2A}{\sigma^2}(I_k \cos(\hat{\phi}_k) + Q_k \sin(\hat{\phi}_k))\right) \lessgtr \gamma'. \tag{5}$$

Taking the logarithm operation on both sides of (5), we get the log-likelihood ratio test as

$$L(\mathbf{y}) = \sum_{k=1}^{N_{nc}} \ln\left(\cosh\left(\frac{2A}{\sigma^2}(I_k \cos(\hat{\phi}_k) + Q_k \sin(\hat{\phi}_k))\right)\right), \tag{6}$$

where $L(\mathbf{y}) = \ln(\Lambda'(\mathbf{y}))$. Now, we must replace the ML estimate of the phase assuming that ϕ_k can take different values for different time instants. In this case, the ML estimate is the arctangent discriminator [8], [9] as $\hat{\phi}_k = \mathrm{atan}(Q_k/I_k)$. By replacing the unknown phase and making some simplifications, the optimal detector is expressed as follows,

$$L(\mathbf{y}) \quad = \quad \sum_{k=1}^{N_{nc}} \left[\ln\left(\cosh\left(\frac{2A|y_k|}{\sigma^2}\right)\right)\right] \lessgtr \gamma, \tag{7}$$

where $\gamma = \ln(\gamma')$. Nevertheless, it is not desirable to use this detector because the relationship between the amplitude A and the noise power σ^2 must be known a priori and, in practice, this is normally not possible. To circumvent this impairment, some approximations can be made to the function $\ln(\cosh(x))$.

On the one hand, by Taylor's formula, an approximation of $\cosh(x)$ for small values of x is $1 + x^2/2$. Moreover, the Taylor series expansion of the logarithm function is $\ln(x+1) \approx x$. Combining the two series, we obtain $\ln(\cosh(x)) \approx x^2/2$. Thus for small values of x and including all the constant terms in the threshold, the resulting detector is well approximated by the conventional NPDI technique as

$$Z_{\text{NPDI}} = \sum_{k=1}^{N_{nc}} |y_k|^2 \lessgtr \gamma_{Z_{\text{NPDI}}}. \quad (8)$$

On the other hand, for large values of x, the $\cosh(x)$ can be approximated by $e^{|x|}/2$. Taking the logarithm function, the approximation of the function $\ln(\cosh(x))$ is $|x| - \ln(2)$. Thereby, if x is large, the approximation of the resulting detector leads to the NPDI_{nq} technique since the constant terms can be incorporated into the threshold as

$$Z_{\text{NPDI}_{\text{nq}}} = \sum_{k=1}^{N_{nc}} |y_k| \lessgtr \gamma_{Z_{\text{NPDInq}}}. \quad (9)$$

IV. Detection Threshold

The signal detection boils down to the comparison of the metric Z of a PDI technique with a detection threshold to distinguish either the satellite is considered to be in view or not. The detection threshold is affected by the individual probability of false alarm (P_{fa}) as

$$P_{fa} = 1 - \text{cdf}_Z(\gamma; H_0), \quad (10)$$

where $\text{cdf}_Z(\gamma; H_0)$ is the cumulative density function of the metric of a PDI technique under the condition H_0. However, the definition of the detection threshold is usually set fixing a value of global probability of false alarm (P_{FA}). The P_{FA} depends on P_{fa} as

$$P_{FA} = 1 - (1 - P_{fa})^L, \quad (11)$$

where L is the number of independent trial points of Doppler frequency and time-delay.

A. Detection threshold for the NPDI technique

The detection threshold for the NPDI technique can be defined in closed-form as

$$\gamma_{Z_{\text{NPDI}}} = \text{cdf}_{\text{X}}^{-1}(1 - P_{fa}, 2N_{nc}; H_0), \quad (12)$$

where $\text{cdf}_{\text{X}}^{-1}$ is the inverse cumulative distribution function of a chi-square with $2N_{nc}$ degrees of freedom.

B. Detection threshold for the NPDI_{nq} technique

The definition of a threshold for the NPDI_{nq} technique is more complicated than the determination of a threshold for the NPDI technique since the cdf of the NPDI_{nq} metric is not known in closed-form. This occurs because the cdf of the NPDI_{nq} metric is composed by the sum of N_{nc} independent Rayleigh random variables and the calculation of this cdf is far from being trivial. The approach used in the literature to define the detection threshold for the NPDI_{nq} technique is by invoking the CLT because the NPDI_{nq} metric asymptotically converges to a Gaussian distribution for large values of N_{nc} [10].

However, although the CLT approximation provides a tight fit in the central part of the statistical distribution, it is often too loose at the tail, where false alarm probabilities need to be calculated. For this reason, in this subsection, we improve the accuracy of the CLT approximation used to estimate the pdf of the NPDI_{nq} metric. To do so, a non-linear transformation can be applied to enhance the convergence speed of the NPDI_{nq} metric to a Gaussian distribution with respect to N_{nc}. In this case, if the metric of NPDI_{nq} is raised to an appropriate fractional exponent, the pdf of the NPDI_{nq} metric may converge faster to a Gaussian distribution. To define the detection threshold for the NPDI_{nq} technique following this approach, the mean and the variance of the NPDI_{nq} metric raised to a fractional exponent must be calculated. To do this, we suppose that Y is a random variable, which is raised to β as

$$T = Y^{\beta}. \quad (13)$$

The mean and the variance of T, denoted by μ_T and σ_T^2, can be calculated using a Taylor series, which results in [11]

$$\mu_T \approx \mu_Y^{\beta} \left[1 + \frac{1}{2}\beta(\beta - 1)\frac{\sigma_Y^2}{\mu_Y^2}\right], \quad (14)$$

$$\sigma_T^2 \approx \beta^2 \frac{\sigma_Y^2}{\mu_Y^{2(1-\beta)}}, \quad (15)$$

where μ_Y is the mean of Y and σ_Y^2 is its variance. Applying (14) and (15) in our problem, the mean and the variance of the NPDI_{nq} metric under H_0 raised to the power of β are given by (16) and (17), respectively

$$\mu_{\text{NPDI}_{\text{nq}}^{\beta}|H_0} \approx \mu_{\text{NPDI}_{\text{nq}}|H_0}^{\beta} \left[1 + \frac{1}{2}\beta(\beta - 1)\frac{\sigma_{\text{NPDI}_{\text{nq}}|H_0}^2}{\mu_{\text{NPDI}_{\text{nq}}|H_0}^2}\right], \quad (16)$$

$$\sigma_{\text{NPDI}_{\text{nq}}^{\beta}|H_0}^2 \approx \beta^2 \frac{\sigma_{\text{NPDI}_{\text{nq}}|H_0}^2}{\mu_{\text{NPDI}_{\text{nq}}|H_0}^{2(1-\beta)}}, \quad (17)$$

where $\mu_{\text{NPDI}_{\text{nq}}|H_0}$ and $\sigma_{\text{NPDI}_{\text{nq}}|H_0}^2$ are the mean and the variance of the NPDI_{nq} metric under H_0, respectively, which are given by

$$\mu_{\text{NPDI}_{\text{nq}}|H_0} = N_{nc}\sigma\frac{\sqrt{\pi}}{2}, \quad (18)$$

$$\sigma_{\text{NPDI}_{\text{nq}}|H_0}^2 = N_{nc}\sigma^2\left(1 - \frac{\pi}{4}\right). \quad (19)$$

Finally, the threshold for the NPDI_{nq} technique is defined as

$$\gamma_{Z_{\text{NPDInq}}} \approx \text{cdf}_{\text{N}}^{-1}\left(1 - P_{fa}, \mu_{\text{NPDI}_{\text{nq}}^{\beta}|H_0}, \sigma_{\text{NPDI}_{\text{nq}}^{\beta}|H_0}; H_0\right)^{\frac{1}{\beta}}, \quad (20)$$

where $\text{cdf}_{\text{N}}^{-1}$ is the inverse cumulative distribution function of a Gaussian distribution.

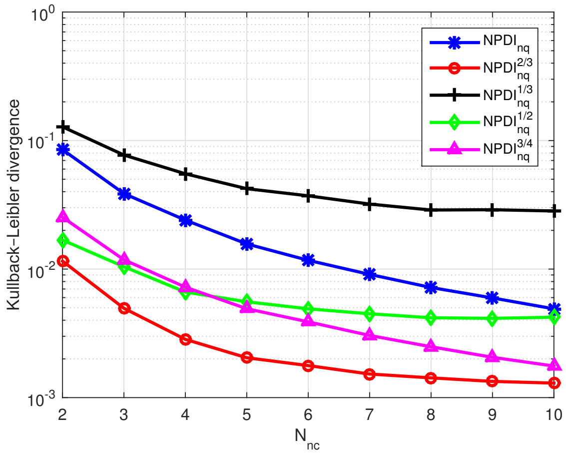

Fig. 1. Kullback-Leibler divergence between the empirical pdf and the Gaussian approximation for different fractional exponents.

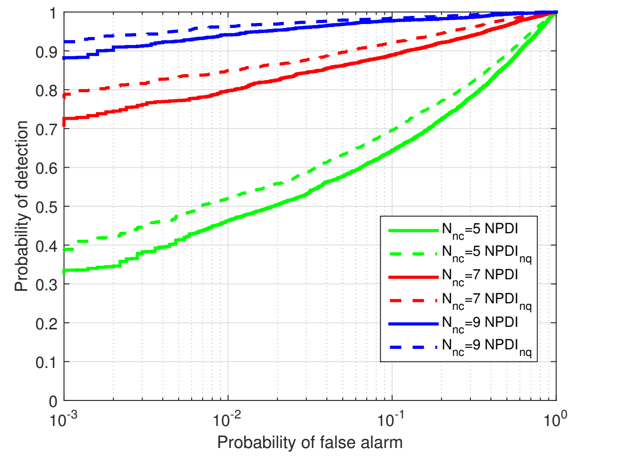

Fig. 2. ROC comparison between the NPDI and $\mathrm{NPDI_{nq}}$ techniques for coherent integration time of 100 ms and $C/N_0 = 20$ dBHz.

Optimum value of β: The problem is to find the optimum fractional exponent to improve the convergence speed of the $\mathrm{NPDI_{nq}}$ metric to a Gaussian distribution for small values of N_{nc}, which are the typical values used in HS-GNSS receivers to acquire weak signals. In order to determine which is the best value of β to enhance the convergence speed of the $\mathrm{NPDI_{nq}}$ metric, we use the Kullback-Leibler (KL) distance or divergence. The KL divergence is a measure of the distance between two pdfs as

$$D_{KL}(P||Q) = \sum_{i=1}^{M} P(i) \ln\left(\frac{P(i)}{Q(i)}\right), \tag{21}$$

where $i = 1, \ldots, M$, M is the number of points of the pdf, $P(i)$ is the pdf of the $\mathrm{NPDI_{nq}}$ metric and $Q(i)$ is the Gaussian approximation of the $\mathrm{NPDI_{nq}}$ metric.

The KL distance is computed by using different fractional exponent values i.e. raising the term $\mathrm{NPDI_{nq}}$ to $\beta = 1/3, 1/2, 2/3, 3/4$, and 1 in Fig. 1. The case of $\beta = 1$ corresponds to the CLT approximation used in the literature. The result shows that the minimum error between the empirical pdf and the Gaussian approximation is obtained for $\beta = 2/3$. As we will see, the use of $\beta = 2/3$ in the $\mathrm{NPDI_{nq}}$ outperforms the accuracy of the computation of the detection threshold compared to the conventional CLT approximation. We have used the fractional exponent of 2/3 in the simulation on purpose since it is known the convergence speed of a chi-squared random variable to a Gaussian distribution can be improved using the fractional exponent of 1/3 [11]. If a Rayleigh distribution is squared, this distribution becomes a chi-square with 2 degrees of freedom. Then, intuitively, if the $\mathrm{NPDI_{nq}}$ metric is raised to 2/3, its convergence to a Gaussian distribution will be faster, as we confirm in Fig. 1.

V. Simulation Results

The simulation results present the Receiver Operating Characteristic (ROC) of the NPDI and $\mathrm{NPDI_{nq}}$ detectors, and the statistical characterization of the $\mathrm{NPDI_{nq}}$ metric. Simulations are based on a Galileo E1BC signal [12]. The acquisition of the signal is performed by the application of a HS-GNSS receiver, which utilizes the double-FFT algorithm [13], [1]. This algorithm provides an efficient signal acquisition using a long coherent integration time and assuming some assistance information about the Doppler of the satellite. The approach used in the simulations is to adopt a small value of N_{nc} and a long coherent integration time, which is the typical way to acquire weak signals in HS-GNSS receivers.

Fig. 2 compares the ROC (i.e. the detection probability versus the global false alarm probability) of the NPDI and $\mathrm{NPDI_{nq}}$ detectors for a coherent integration time of 100 ms, carrier-to-noise ratio (C/N_0) equal to 20 dBHz and $N_{nc} = 5, 7$ and 9. The results show that using a small value of N_{nc} the $\mathrm{NPDI_{nq}}$ technique outperforms the NPDI technique in terms of signal detection. This fact occurs because the term inside the function cosh in (7) is not a small number since the signal can be detected by applying a few non-coherent integrations. Therefore, the optimal detector is more accurately approximated by the $\mathrm{NPDI_{nq}}$ technique providing a gain over the NPDI technique. This is an important result because GNSS receivers usually use the NPDI technique to acquire signals. However, as it can be seen in Fig. 2, the use of the $\mathrm{NPDI_{nq}}$ technique leads to a better performance to detect signals in the context of HS-GNSS receivers.

Fig. 3 illustrates the P_{fa} of the $\mathrm{NPDI_{nq}}$ metric, the CLT approximation and the new approximation proposed herein by introducing the fractional exponent for different numbers of N_{nc}. The CLT approximation provides a significant error with respect to the empirical value of P_{fa}, especially for small values of P_{fa}. The value of the P_{fa} obtained analytically for $\mathrm{NPDI_{nq}}$ under H_0 using the Gaussian approximation after having applied the $\beta = 2/3$ exponent is much more accurate in the tail region. From this result, we can conclude that the approximation, which uses the fractional exponent, allows us

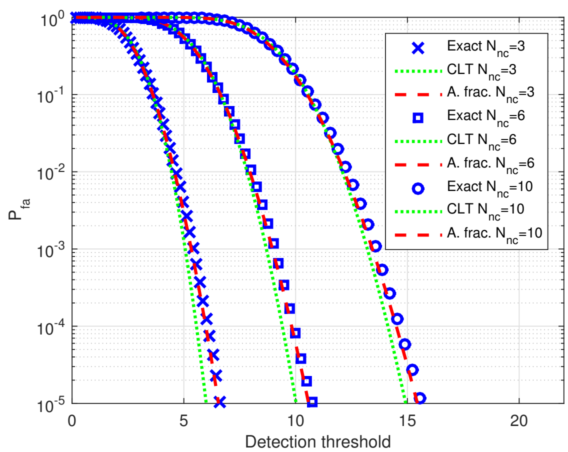

Fig. 3. Empirical and approximated P_{fa} of the NPDI_{nq} technique for $N_{nc} = 3, 6$, and 10 (in the legend, A. frac. refers to the approximation by using the fractional exponent).

to define a more accurate detection threshold for the NPDI_{nq} technique.

VI. Conclusions

Based on the GLRT approach, this paper has derived the optimal PDI technique for HS-GNSS receivers. This PDI technique is difficult to implement in practice since the amplitude of the signal must be known a priori. Due to this fact, two approximations of this detector are used leading to the NPDI and NPDI_{nq} techniques. We have shown that the NPDI_{nq} technique outperforms the NPDI technique, which is the most common technique used to detect weak signals in HS-GNSS receivers. In addition to this, a new statistical characterization of the NPDI_{nq} metric has been proposed. We have obtained the optimum value of the exponent of the sum of Rayleigh variables by minimizing the Kullback-Leibler distance to a target Gaussian distribution. Finally, we have shown that this transformation allows us to predict the relation between the probability of false alarm and the detection threshold with extraordinary accuracy.

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