STATISTICAL CHARACTERIZATION OF THE OPTIMAL DETECTOR FOR A SIGNAL WITH TIME-VARYING PHASE BASED ON THE EDGEWORTH SERIES

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ABSTRACT

This paper focuses on approximating the false alarm and detection probabilities of the optimal non-coherent detector for a signal, which contains a constant amplitude and unknown phase, corrupted by Gaussian noise. Several closed-form approximations of these probabilities are obtained using different truncations of the Edgeworth series and the Central Limit Theorem (CLT). The accuracy of the different approximations is contrasted to the performance of the optimal non-coherent detector revealing that the best approximation corresponds to the Edgeworth expansion using the longest series, which offers a great precision. The CLT approximation is not accurate enough to predict the performance of the optimal detector. The closed-form expression based on the Edgeworth series allows us to set a detection threshold for a false alarm probability value and obtain the detection probability of the detector with extreme accuracy.

Index Terms— CLT, detection threshold, Edgeworth series, post-detection integration techniques, ROC curves.

1. INTRODUCTION

Many technologies such as Global Navigation Satellite Systems (GNSS), Cognitive Radio (CR), and radar, require detecting weak signals with power levels below the noise level. High-sensitivity GNSS receivers need to acquire weak signals to be able to provide an estimation of its location, particularly in indoor or urban environments [1, 2]. CR applications detect these signals to know the availability of frequency bands [3]. Radar systems also use detection techniques to obtain the position of vehicles and atmospheric research [4]. These technologies are clear examples in which the application of weak signal detection techniques is of paramount importance.

In weak reception conditions, the receiver is not usually able to detect the signals since they arrive highly attenuated owing to the presence of obstacles in the path between the transmitter and the receiver. In this situation, the receiver must apply non-coherent detection techniques or post-detection integration techniques to detect these weak signals. The problem of detecting a weak signal, which includes a constant amplitude and unknown phase that varies during the time immersed in additive white Gaussian noise, has been widely studied and the optimal non-coherent detector is well-known. Nevertheless, the drawback of this detector is that its detection and false alarm probabilities are unknown in closed-form. This optimal detector is often approximated by the square law detector, which is a good approximation of the optimal detector, especially for really small values of Signal-to-Noise Ratio (SNR) [4, 5, 6]. Moreover, the square law detector is easy to implement in a receiver and it also has closed-form expressions for its detection and false alarm probabilities.

The advantages of the square-law detector have lead most receivers to use this detector to acquire weak signals. However, for relatively large SNR values, the approximation of this detector becomes less accurate causing a degradation performance with respect to the optimal detector. These values of SNR are easy to find in detection problems where the signal must be discriminated among several samples of noise and detected applying a small number of non-coherent combinations. In the high SNR regime, the optimal detector can be approximated by the linear detector [4].

Nonetheless, the exact expressions of the false alarm and detection probabilities for the linear detector are completely unknown. Two analysis of this detector are found in [5, 7]. Although theoretical analysis of the linear and square law detectors have been widely carried out in the literature, the literature still lacks a theoretical analysis of the optimal non-coherent detector, as far as the authors know. This theoretical analysis is really important since it would predict the performance of the optimal non-coherent detector in any SNR region and set a detection threshold from a false alarm probability value.

For this reason, the purpose of this paper is to provide a closed-form expression of the false alarm and detection probabilities for the optimal non-coherent detector. These expressions are obtained by applying the Edgeworth series, which offer an excellent approximation of the sum of random variables. Moreover, the accuracy of different truncations of these series is compared to the accuracy provided by the Central Limit Theorem (CLT), revealing a clear advantage in favour of the Edgeworth series.

2. SIGNAL MODEL

The detection of weak signals is a statistical hypothesis problem and it is usually solved by using the detection theory [8, 9]. The receiver usually discriminates between the hypothesis H_0, the signal is absent, and the hypothesis H_1, the signal is present, as

- Under H_0: $x_k = \omega_k$ is a complex additive white Gaussian noise with zero-mean and variance σ^2 .
- Under H_1: $x_k = Ae^{j\phi_k} + \omega_k$ corresponds to the signal plus complex additive white Gaussian noise with zero-mean and variance σ^2,

where A is a constant affected by an unknown phase ϕ_k and x_k is the received signal in the time instant k. The discrimination between the two hypotheses is carried out by setting a signal detection threshold. However, there many situations where the noise level does not allow the receiver to detect the signal. In these circumstances, the receiver must apply non-coherent detection techniques to be able to acquire the weak signal. The optimal non-coherent detector is obtained by using the Bayesian approach for our signal model assuming that the phase ϕ_k is a uniform random variable on the interval $(0, 2\pi]$. This result is well-known in the literature [4, 5] and is given by

$$Y = \sum_{k=1}^{N_{nc}} \ln\left[I_0\left(\frac{2A|x_k|}{\sigma^2}\right)\right] \lessgtr \gamma, \tag{1}$$

where $k = 1, ..., N_{nc}$, N_{nc} is the number of non-coherent combinations, γ is the detection threshold and I_0 is the modified Bessel function of order 0. The distribution of the random variable Y is completely unknown since it is composed by the sum of N_{nc} independent unknown distributions. The distribution of the metric Y provides highly desirable information about the performance of the detector in (1) since it allows us to obtain the false alarm and detection probabilities, which are defined as

$$P_{fa} = 1 - \text{cdf}_Y(\gamma; H_0) \tag{2}$$

$$P_d = 1 - \text{cdf}_Y(\gamma; H_1), \tag{3}$$

where $\text{cdf}_Y(\gamma; H_0)$ and $\text{cdf}_Y(\gamma; H_1)$ are the cumulative density function under H_0 and H_1 of the metric Y, respectively.

3. APPROXIMATION OF DETECTION AND FALSE ALARM PROBABILITIES FOR THE OPTIMAL DETECTOR

Closed-form expressions of the detection and false alarm probabilities become necessary to set an appropriate detection threshold or to be able to predict the performance of a detector. In our problem, these probabilities require the knowledge about the cdf of Y under H_0 and H_1. However, closed-form expressions of these cdfs are not known owing to the complexity introduced by the sum of N_{nc} independent random variables, which use modified Bessel function.

In this situation, approximations of the cdfs of Y are needed to be able to compute the probabilities of interest. A simple approximation of the sum of distributions involves the use of the CLT theorem because the variable Y asymptotically converges to a Gaussian distribution for large values of N_{nc}. However, if the N_{nc} value is not large enough, the CLT does not offer an acceptable approximation, particularly at the tail region, where the probabilities of interest are often calculated.

One way to reduce the error introduced by the CLT approximation is by exploiting the Edgeworth series, which use some coefficients that depend on the moments of the variable Y [10, 11, 12, 13]. Another approach consists in applying the saddle-point approximation, which could offer even better accuracy than the Edgeworth series. Nevertheless, the saddle-point approximation requires the prior knowledge about the moment-generating function for the distribution of interest [14]. Unfortunately, this function is unknown for the problem at hand. For this reason, the best option to estimate the distribution of the variable Y is by using the Edgeworth series.

3.1. Edgeworth series

Edgeworth series are an indispensable tool to obtain an accurate approximation of the probability density function (pdf) and cdf for a random variable, which has been obtained from summing several independent random variables. These series provide us some clues to enhance the CLT approximation by introducing some terms that depend on Hermite polynomials and the moments of the random variable. More precisely, Edgeworth series are a particular case of the Gram-Charlier Type A series, which are defined as

$$f_{GC}(\tilde{Y}) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{\tilde{Y}^2}{2}} \left[1 + \sum_{n=3}^{\infty} \frac{C_n}{n!} H_n\left(\tilde{Y}\right)\right], \tag{4}$$

$$F_{GC}(\tilde{Y}) = \Phi(\tilde{Y}) - \frac{1}{\sqrt{2\pi}} e^{-\frac{\tilde{Y}^2}{2}} \left[\sum_{n=3}^{\infty} \frac{C_n}{n!} H_{n-1}\left(\tilde{Y}\right)\right], \tag{5}$$

where f_{GC} and F_{GC} are the Gram-Charlier Type A series approximation for the pdf and cdf, respectively, $\tilde{Y} = \frac{Y-\mu_Y}{\sigma_Y}$, μ_Y and σ_Y are the mean and the standard deviation of the variable Y, which are obtained from evaluating a numerical integral due to the complexity of the variable Y, $\Phi(\tilde{Y}) = \int_{-\infty}^{\tilde{Y}} \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{\lambda^2}{2}} d\lambda$, $H_n(\tilde{Y})$ are the Hermite polynomials, which are given by

$$H_n(\tilde{Y}) = (-1)^n e^{\frac{\tilde{Y}^2}{2}} \frac{\partial^n}{\partial \tilde{Y}^n} e^{\frac{-\tilde{Y}^2}{2}}. \tag{6}$$

The coefficients C_n can be expressed as

$$C_n = \int_{-\infty}^{\infty} H_n(\tilde{Y}) \text{pdf}_Y(\tilde{Y}) d\tilde{Y}, \tag{7}$$

where $\text{pdf}_Y(\tilde{Y})$ is the pdf of the random variable Y. After some straightforward, but tedious manipulations, it is found that

$$C_3 = \frac{\mu_{Y,3} - 3\mu_{Y,1}\mu_{Y,2} + 2\mu_{Y,1}^3}{\sigma_Y^3}, \tag{8}$$

$$C_4 = \frac{\mu_{Y,4} - 4\mu_{Y,1}\mu_{Y,3} + 6\mu_{Y,1}^2\mu_{Y,2} - 3\mu_{Y,1}^4}{\sigma_Y^4} - 3, \tag{9}$$

$$\begin{aligned} C_5 = &\frac{\mu_{Y,5} - 5\mu_{Y,4}\mu_{Y,1} + 10\mu_{Y,3}{\mu_{Y,1}}^2 - 10\mu_{Y,2}{\mu_{Y,1}}^3}{\sigma_Y^5} \\ &+ \frac{4{\mu_{Y,5}}^5}{\sigma_Y^5} - 10\frac{\mu_{Y,3} - 3\mu_{Y,1}\mu_{Y,2} + 2{\mu_{Y,1}}^3}{{\sigma_Y}^3}, \end{aligned} \tag{10}$$

$$\begin{aligned} C_6 = &\frac{\mu_{Y,6} - 6\mu_{Y,5}\mu_{Y,1} + 15\mu_{Y,4}{\mu_{Y,1}}^2 - 20{\mu_{Y,1}}^3\mu_{Y,3}}{\sigma_Y^6} \\ &+ \frac{15\mu_{Y,2}{\mu_{Y,1}}^4 - 5{\mu_{Y,1}}^6}{\sigma_Y^6} + 30 \\ &- 15\frac{\mu_{Y,4} - 4\mu_{Y,1}\mu_{Y,3} + 6\mu_{Y,1}^2\mu_{Y,2} - 3\mu_{Y,1}^4}{\sigma_Y^4}, \end{aligned} \tag{11}$$

where $\mu_{Y,p} = \text{E}[Y^p]$ is the p-th moment of the random variable under analysis, which has been calculated through many tedious manipulations. This result is shown at the top of next page. The $\mu_{Y,p}$ moments depend on the moments of $\ln\left[I_0\left(\frac{2A|x_k|}{\sigma^2}\right)\right]$ variable, which are denote as $\mu_{x,l} = \text{E}\left[\left(\ln\left[I_0\left(\frac{2A|x_k|}{\sigma^2}\right)\right]\right)^l\right]$. Because of the Rayleigh or Rice nature of $|x_k|$ for the hypotheses H_0 and H_1, respectively, the moments can be computed numerically for H_0

$$\mu_{x,l,H_0} = \int_0^{\infty} \left(\ln\left[I_0\left(\frac{2A|x_k|}{\sigma^2}\right)\right]\right)^l \text{pdf}_{|x_k|}(|x_k|; H_0) d|x_k|, \tag{18}$$

and for H_1

$$\mu_{x,l,H_1} = \int_0^{\infty} \left(\ln\left[I_0\left(\frac{2A|x_k|}{\sigma^2}\right)\right]\right)^l \text{pdf}_{|x_k|}(|x_k|; H_1) d|x_k|. \tag{19}$$

$$
\begin{aligned}
\mu_{Y,1} =& N_{nc}\mu_{x,1}, && (12)\\
\mu_{Y,2} =& N_{nc}(\mu_{x,2} + (N_{nc}-1)\mu_{x,1}^2), && (13)\\
\mu_{Y,3} =& N_{nc}(\mu_{x,3} + (N_{nc}-1)(3\mu_{x,2}\mu_{x_k,1} + (N_{nc}-2)\mu_{x,1}^3)), && (14)\\
\mu_{Y,4} =& N_{nc}(\mu_{x,4} + (N_{nc}-1)(4\mu_{x,3}\mu_{x,1} + 3\mu_{x,2}^2 + (N_{nc}-2)(6\mu_{x,2}\mu_{x,1}^2 + (N_{nc}-3)\mu_{x,1}^4))), && (15)\\
\mu_{Y,5} =& N_{nc}\,(\mu_{x,5} + (N_{nc}-1)(5\mu_{x,4}\mu_{x,1} + 10\mu_{x,3}\mu_{x,2} + (N_{nc}-2)(10\mu_{x,3}{\mu_{x,1}}^2 + 15{\mu_{x,2}}^2\mu_{x,1} + (N_{nc}-3)(10{\mu_{x,1}}^3\mu_{x,2} \\
& + (N_{nc}-4){\mu_{x,1}}^5)))), && (16)\\
\mu_{Y,6} =& N_{nc}(\mu_{x,6} + (N_{nc}-1)(6\mu_{x,5}\mu_{x,1} + 10{\mu_{x,3}}^2 + 15\mu_{x,4}\mu_{x,2} + (N_{nc}-2)(15\mu_{x,4}\,{\mu_{x,1}}^2 + 15{\mu_{x,2}}^3 + 60\mu_{x,2}\mu_{x,3}\mu_{x,1} \\
& + (N_{nc}-3)(20\mu_{x,3}{\mu_{x,1}}^3 + 45{\mu_{x,1}}^2{\mu_{x,2}}^2 + (N_{nc}-4)(15\mu_{x,2}{\mu_{x,1}}^4 + (N_{nc}-5){\mu_{x,1}}^6))))). && (17)
\end{aligned}
$$

Series	L	C_n^*
E. 1	3	C_3
E. 2	3,4,6	$C_3, C_4, 10C_3^2$
E. 3	3,4,6, 5,7,9	$C_3, C_4, 10C_3^2$, $C_5, 35C_4C_3, 280C_3^3$
E. 4	3,4,6, 5,7,9, 8,10,12	C_3, C_4, C_6 $C_5, 35C_4C_3, 280C_3^3$ $35C_4^2 + 56C_5C_3, 2100C_3^2C_4, 15400C_3^4$

Table 1. Relationship between the group of terms L and the coefficients C_n.

Although the series in (4) and (5) decrease as $1/n!$ in the coefficients, they suffer from having poor convergence properties, which can cause an inaccurate estimation of the pdf of interest. This problem is circumvented by taking a specific grouping of terms that guarantees the convergence of the series expansion. These groupings of terms lead to the approximations known as Edgeworth series, which are given by (20) and (21)

$$
f_E(\tilde{Y}) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{\tilde{Y}^2}{2}} \left[1 + \sum_{n\in L} \frac{C_n^*}{n!} H_n\left(\tilde{Y}\right) \right], \qquad (20)
$$

$$
F_E(\tilde{Y}) = \Phi(\tilde{Y}) - \frac{1}{\sqrt{2\pi}} e^{-\frac{\tilde{Y}^2}{2}} \left[\sum_{n\in L} \frac{C_n^*}{n!} H_{n-1}\left(\tilde{Y}\right) \right], \qquad (21)
$$

where f_E and F_E are the Edgeworth series approximation for the pdf and the cdf, respectively. There are different group terms L that guarantee the convergence of the series. These group of terms are affected by the coefficients C_n^*. The relationship between the group of terms L and the coefficients C_n^* is shown in Table 1. The more coefficients are included in the Edgeworth series approximation, the more accurate the approximation tends to be. Nevertheless, if no coefficients are added, the Edgeworth series is the same as the CLT approximation.

Taking into account the expressions in (20) and (21), it is possible to obtain the approximation of the $\mathrm{cdf}_Y(\gamma; H_0)$ and $\mathrm{cdf}_Y(\gamma; H_1)$ depending whether the moments of the variable Y have been computed using (18) for H_0 or using (19) for H_1. Finally, from the estimation of $\mathrm{cdf}_Y(\gamma; H_0)$ and $\mathrm{cdf}_Y(\gamma; H_1)$, we can obtain the false alarm and detection probabilities of the optimal non-coherent detector using the expressions in (2) and (3), respectively.

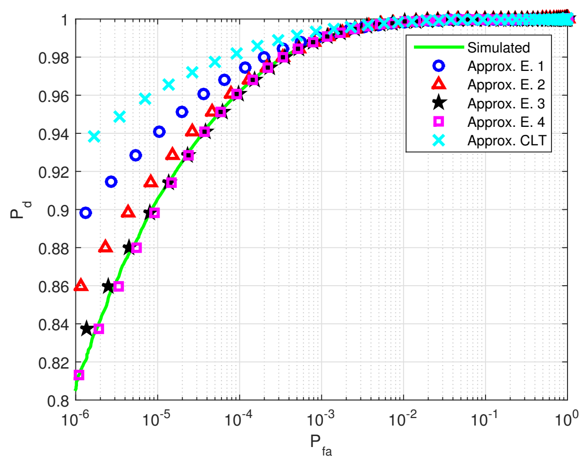

Fig. 1. ROC curve approximations for the performance of the optimal detector using the Edgeworth series and the CLT approximation for parameters: $A = 1.7, \sigma = 1$, and $N_{nc} = 10$.

4. SIMULATION RESULTS

Simulation results are based on comparing the performance of the optimal detector in (1) obtained through the Monte Carlo simulations with the theoretical approximations proposed herein. Fig. 1 illustrates ROC (Receiver Operating Characteristic) curves, which show the P_d vs. P_{fa}, for the optimal non-coherent detector, the approximations obtained using the different Edgeworth series and the CLT approximation. The result shows that the CLT approximation is a very inaccurate approximation, particularly for small values of P_{fa}. Nevertheless, the Edgeworth series approximation reduces the error offered by the CLT approximation. The more coefficients we add to the series, the smaller the error between the simulated ROC curve and the theoretical one. From this result, we can conclude that the E.4 approximation of the Edgeworth series defined in the Table 1 is the most accurate approximation and it allows us to predict the performance of the optimal detector even for small values of P_{fa}, which are the most common values used in the receivers.

Fig. 2 shows the P_d vs. SNR for the optimal non-coherent detector and the E.4 approximation obtained from the Edgeworth series. The result shows that the E.4 approximation is a very good fit and it is able to predict the detection probability of the optimal detector for any value of N_{nc} and SNR. This is an important result since it

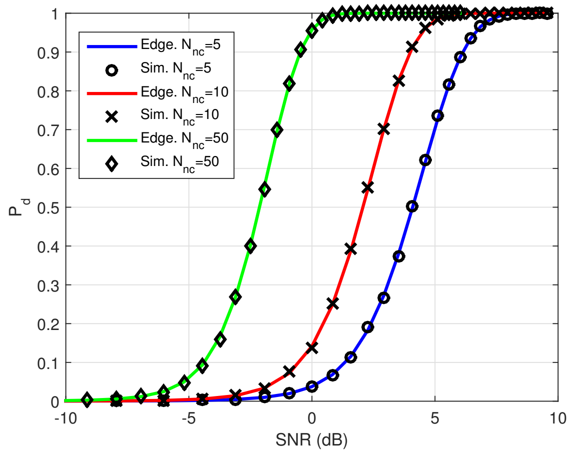

Fig. 2. P_d vs. SNR with $N_{nc} = 5, 10, 50$ and $P_{fa} = 1e-4$ for the E.4 approximation and the optimal non-coherent detector obtained from simulations.

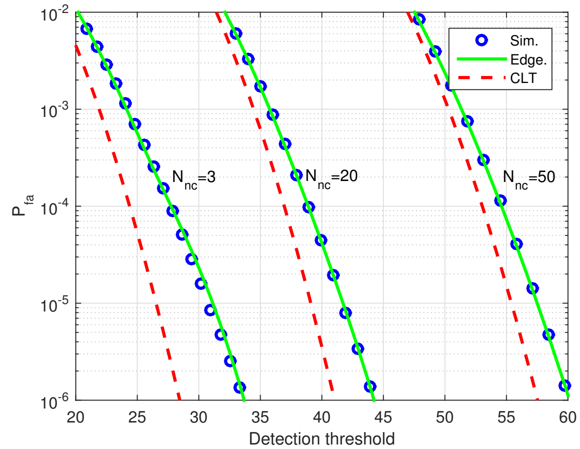

Fig. 3. P_{fa} vs. detection threshold with $N_{nc} = 3, 20$ and 50 for the E.4 and CLT approximations and the optimal non-coherent detector obtained from simulations.

provides us prior knowledge about the P_d of this detector.

Fig. 3 illustrates the error between the P_{fa} of the optimal detector with the E.4 and CLT approximations. The use of the Edgeworth series is preferable since it is a more effective approximation than the CLT, particularly at the tail region. The Edgeworth series allow us to set an extremely accurate detection threshold to distinguish if the signal is present or absent. The error introduced by the CLT is really significant, especially for small values of P_{fa}, which are typically implemented in receivers to avoid false alarm problems.

5. CONCLUSIONS

This paper has proposed different approximations of the false alarm and detection probabilities for the optimal non-coherent detector, which are based on the Edgeworth series and the CLT. Simulation results prove that the approximation E.4 of the Edgeworth series, which introduces a larger number of coefficients, is the most effective one. This approximation allows us to predict the performance of the detector and set an accurate signal detection threshold. The Edgeworth approximations E.1, E.2 and E.3, which use less coefficients than the E.4 approximation, exhibit a larger error. We can conclude that, in this case, the more coefficients the Edgeworth series uses, the more accurate the approximation tends to be. The most inaccurate approximation is the one that corresponds to the CLT, which provides a poor accuracy, especially for small values of false alarm probability.

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